M2REG-INVM-2324-ASM-SET 2-MATH
Suggested solutions

1. (a) $A^2 = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ 1A

$$\alpha A^2 + \beta A = I$$

$$\begin{pmatrix} 4\alpha + 2\beta & 0 \\ 0 & 9\alpha - 3\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We have $4\alpha + 2\beta = 1$ and $9\alpha - 3\beta = 1$. 1M

Solving, we have $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{6}$. 1A

(b) $$\frac{1}{6}A^2 + \frac{1}{6}A = I$$

$A\left[\frac{1}{6}A + \frac{1}{6}I\right] = \left[\frac{1}{6}A + \frac{1}{6}I\right]A = I$ 1M

$A^{-1} = \frac{1}{6}(A + I)$ 1M

$= \frac{1}{6}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ 1A

(c) $ABA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$B = A^{-1}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}A^{-1}$ 1M

$= \frac{1}{36}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$

$= \frac{1}{36}\begin{pmatrix} 3 & 6 \\ -6 & -8 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$

$= \frac{1}{36}\begin{pmatrix} 9 & -12 \\ -18 & 16 \end{pmatrix}$ 1A

2. (a) $R = \frac{1}{3}A - \frac{1}{3}B$

$= \frac{1}{3}\begin{pmatrix} 9 & 3 \\ -39 & -12 \end{pmatrix}$

$= \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}$ 1A

(b) $R^2 = \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} = \begin{pmatrix} -4 & -1 \\ 13 & 3 \end{pmatrix}$ 1A

$R^3 = \begin{pmatrix} -4 & -1 \\ 13 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$R(R^2) = (R^2)R = I$

Thus, $R^2 = R^{-1}$. 1

(c) $R^{31} = (R^3)^{10}R$ 1M

$= R$

$= \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}$ 1A

3. (a) $AB = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$ 1M

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ 1M

Thus, $A^{-1} = B$ and $B^{-1} = A$. 1

(b) $ACB = P$

$C = A^{-1}PB^{-1}$ 1M

$= A^{-1}PA$

$C^n = (A^{-1}PA)^n$

$= A^{-1}P^nA$ 1M

$= BP^nA$ 1

(c) $BPA = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -4 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ 1A

$BP^{20}A = (BPA)^{20}$

$= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}^{20}$

$= \begin{pmatrix} 2^{20} & 0 \\ 0 & 4^{20} \end{pmatrix}$ 1A

$(BP^{20}A)^{-1} = \dfrac{1}{2^{20}4^{20}} \begin{pmatrix} 4^{20} & 0 \\ 0 & 2^{20} \end{pmatrix}^T$ 1M

$= \begin{pmatrix} 2^{-20} & 0 \\ 0 & 4^{-20} \end{pmatrix}$ 1A

4. (a) $A^2=\begin{pmatrix}1&0&0\\2&1&-1\\1&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\2&1&-1\\1&0&-1\end{pmatrix}=\begin{pmatrix}1&0&0\\3&1&0\\0&0&1\end{pmatrix}$

$A^3=\begin{pmatrix}1&0&0\\2&1&-1\\1&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\3&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\5&1&-1\\1&0&-1\end{pmatrix}$ 1M

$A^3-A^2-A+I=\begin{pmatrix}1&0&0\\5&1&-1\\1&0&-1\end{pmatrix}-\begin{pmatrix}1&0&0\\3&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\2&1&-1\\1&0&-1\end{pmatrix}+\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=O$ 1

(b) (i) When $n=1$,

R.H.S. $=A^2+0I=A^2=$ L.H.S.

It is true for $n=1$. 1

Assume $A^{2k}=kA^2+(1-k)I$, where $k\in\mathbb{Z}^+$. 1

$A^{2(k+1)}=A^{2k}A^2$

$=(kA^2+(1-k)I)A^2$ 1

$=k(A^4-A^2)+A^2$

$=kA(A^3-A)+A^2$

$=kA(A^2-I)+A^2$ [by (a)]

$=k(A^3-A)+A^2$

$=k(A^2-I)+A^2$ [by (a)]

$=(1+k)A^2-kI$

It is true for $n=k+1$.

By M.I., it is true $\forall n\in\mathbb{Z}^+$. 1

(ii) $A^{2019}=A^{2(1009)}A$

$=[1009A^2+(1-1009)I]A$ 1M

$=1009\begin{pmatrix}1&0&0\\3&1&0\\0&0&1\end{pmatrix}-1008\begin{pmatrix}1&0&0\\2&1&-1\\1&0&-1\end{pmatrix}$

$=\begin{pmatrix}1&0&0\\3029&1&-1\\1&0&-1\end{pmatrix}$

$\det\left(A^{2019}\right)=\begin{vmatrix}1&-1\\0&-1\end{vmatrix}=-1$

$$((A^{-1})^T)^{2019} = ((A^{2019})^{-1})^T \qquad \text{1M}$$

$$= \frac{1}{-1}\begin{pmatrix} -1 & 0 & 0 \\ 3028 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}^T \qquad \text{1M}$$

$$= \begin{pmatrix} 1 & -3028 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \qquad \text{1A}$$

5. (a) (i) $\begin{pmatrix} \alpha-2 & 2 \\ 1 & \beta-2 \end{pmatrix}\begin{pmatrix} \alpha-3 & 2 \\ 1 & \beta-3 \end{pmatrix}=\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$\begin{pmatrix} \alpha^2-5\alpha+8 & 2(\alpha+\beta-5) \\ \alpha+\beta-5 & \beta^2-5\beta+8 \end{pmatrix}=\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ 1A

We have $\alpha^2-5\alpha+8=4$, $\alpha+\beta-5=0$ and $\beta^2-5\beta+8=4$. 1M

α and β are roots of the equation $x^2-5x+4=0$.

Thus, $\alpha=4$ and $\beta=1$. 1A+1A

(ii) $\det P=\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix}=2$

$P^{-1}=\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix}^T$ 1M

$=\begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{pmatrix}$ 1A

(b) (i) $PQP^{-1}=\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{pmatrix}$

$=\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{pmatrix}$

$=\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ 1A

(ii) $(PQP^{-1})^n=\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^n$

$PQ^nP^{-1}=\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}$ 1M

$Q^n=P^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}P$ 1M

$=\begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$

$=\begin{pmatrix} \frac{1}{2} & -2^n \\ -\frac{1}{2} & 2^{n+1} \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$

$=\begin{pmatrix} 2-2^n & 1-2^n \\ -2+2^{n+1} & -1+2^{n+1} \end{pmatrix}$ 1A

(iii) $S = Q + P^T + xI$

$$= \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

$$= \begin{pmatrix} 4+x & 0 \\ 4 & 4+x \end{pmatrix}$$

When $x = -4$,

$$S = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$$

$$S^2 = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, $S^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $n = 2, 3, 4, \ldots$

$$\sum_{k=1}^{2021}(k+1)S^k = 2S + 3S^2 + 4S^3 + \ldots + 2022S^{2021}$$

$$= 2S$$ 1A

Thus, the claim is agreed. 1A

6. (a) (i) $M(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} x & 0 & x \\ 2x & 0 & 2x \\ -x & 0 & -x \end{pmatrix}$ 1M

$= I + x\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$

We have $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$. 1A

(ii) $A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 1A

$M(x)M(y) = (I + xA)(I + yA)$

$= I + xA + yA + xyA^2$ 1M

$= I + (x + y)A$

$= M(x + y)$ 1A

(iii) $M(1) = [M(1)]^1$

It is true for $n = 1$. 1

Assume $M(k) = [M(1)]^k$, where $k \in \mathbb{Z}^+$. 1M

$M(k+1) = M(k)M(1)$

$= [M(1)]^k M(1)$ 1M

$= [M(1)]^{k+1}$

It is true for $n = k + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(b) (i) $B = M(2) = [M(1)]^2$ 1M

$B^n = [M(1)]^{2n}$

$= M(2n)$

$= \begin{pmatrix} 1+2n & 0 & 2n \\ 4n & 1 & 4n \\ -2n & 0 & 1-2n \end{pmatrix}$ 1A

(ii) Note that $M(2)M(-2) = M(-2)M(2) = M(0) = I$. 1M

$B^{-1} = [M(2)]^{-1}$

$= M(-2)$

$= \begin{pmatrix} -1 & 0 & -2 \\ -4 & 1 & -4 \\ 2 & 0 & 3 \end{pmatrix}$ 1A

7. (a) $$XY = \frac{1}{(\alpha-\beta)(\beta-\alpha)}\begin{pmatrix}\alpha-\beta-k & \alpha-\beta-k \\ k & k\end{pmatrix}\begin{pmatrix}-k & \alpha-\beta-k \\ k & \beta-\alpha+k\end{pmatrix} = \begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}$$ 1A

$$YX = \frac{1}{(\alpha-\beta)(\beta-\alpha)}\begin{pmatrix}-k & \alpha-\beta-k \\ k & \beta-\alpha+k\end{pmatrix}\begin{pmatrix}\alpha-\beta-k & \alpha-\beta-k \\ k & k\end{pmatrix} = \begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}$$ 1A

$$X+Y = \frac{1}{\alpha-\beta}(A-\beta I - A + \alpha I) = I$$ 1A

$$X^2 = X(I-Y) = X - XY = X$$

$$Y^2 = Y(I-X) = Y - YX = Y$$ 1A

(b) $\alpha X + \beta Y = \dfrac{\alpha}{\alpha-\beta}(A-\beta I) + \dfrac{\beta}{\beta-\alpha}(A-\alpha I) = A$ 1

Since $XY = YX = \mathbf{0}$,

$$A^n = (\alpha X + \beta Y)^n$$

$$= \alpha^n X^n + C_1^n(\alpha X)^{n-1}(\beta Y) + C_2^n(\alpha X)^{n-2}(\beta Y)^2 + \ldots + (\beta Y)^n$$ 1M

$$= \alpha^n X^n + \beta^n Y^n$$ 1M

$$= \alpha^n X + \beta^n Y$$ 1

(c) Put $k = 2$, $\alpha = 7$ and $\beta = 1$ such that $\alpha \neq \beta$. 1A

Define $X = \dfrac{1}{7-1}(A-I) = \dfrac{1}{3}\begin{pmatrix}2 & 2 \\ 1 & 1\end{pmatrix}$ and $Y = \dfrac{1}{1-7}\begin{pmatrix}-2 & 4 \\ 2 & -4\end{pmatrix} = \dfrac{1}{3}\begin{pmatrix}1 & -2 \\ -1 & 2\end{pmatrix}$. 1A

$$\begin{pmatrix}5 & 4 \\ 2 & 3\end{pmatrix}^{2004} = 7^{2004}\cdot\frac{1}{3}\begin{pmatrix}2 & 2 \\ 1 & 1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1 & -2 \\ -1 & 2\end{pmatrix}$$ 1M

$$= \begin{pmatrix}\dfrac{2(7^{2004})+1}{3} & \dfrac{2(7^{2004})-2}{3} \\ \dfrac{7^{2004}-1}{3} & \dfrac{7^{2004}+2}{3}\end{pmatrix}$$ 1A

(d) Consider the matrix $\alpha^{-1}X + \beta^{-1}Y$.

$$(\alpha^{-1}X + \beta^{-1}Y)(\alpha X + \beta Y) = X^2 + \frac{\beta}{\alpha}XY + \frac{\alpha}{\beta}YX + Y^2$$ 1M

$$= X + Y = I$$ 1A

Therefore, $A^{-1} = \alpha^{-1}X + \beta^{-1}Y$. 1A

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