M2REG-MI-2324-ASM-SET 2-MATH
Suggested solutions

1. (a) When $n = 1$,

$$\text{L.H.S.} = \sum_{r=2}^{4} \frac{1}{r(r-1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$\text{R.H.S.} = \frac{2+1}{4(2-1)} = \frac{3}{4} = \text{L.H.S.}$$

It is true for $n = 1$. 1

Assume $\displaystyle\sum_{r=2k}^{4k} \frac{1}{r(r-1)} = \frac{2k+1}{4k(2k-1)}$, where $k \in \mathbb{Z}^+$. 1M

$$\begin{aligned}
&\sum_{r=2k+2}^{4k+4} \frac{1}{r(r-1)} \\
&= \sum_{r=2k}^{4k} \frac{1}{r(r-1)} - \frac{1}{2k(2k-1)} - \frac{1}{(2k+1)(2k)} + \frac{1}{4k(4k+1)} + \frac{1}{(4k+1)(4k+2)} \\
&\quad + \frac{1}{(4k+2)(4k+3)} + \frac{1}{(4k+3)(4k+4)} \\
&= \frac{2k+1}{4k(2k-1)} - \frac{(2k+1)+(2k-1)}{2k(2k-1)(2k+1)} + \frac{(4k+2)+4k}{4k(4k+1)(4k+2)} \qquad \text{1M} \\
&\quad + \frac{(4k+4)+(4k+2)}{(4k+2)(4k+3)(4k+4)} \\
&= \frac{2k+1}{4k(2k-1)} - \frac{2}{(2k-1)(2k+1)} + \frac{1}{4k(2k+1)} + \frac{1}{4(2k+1)(k+1)} \\
&= \frac{(2k+1)^2 - 2(4k)}{4k(2k-1)(2k+1)} + \frac{(k+1)+k}{4k(2k+1)(k+1)} \\
&= \frac{2k-1}{4k(2k+1)} + \frac{1}{4k(k+1)} \\
&= \frac{(2k-1)(k+1)+(2k+1)}{4k(2k+1)(k+1)} \\
&= \frac{2k+3}{4(k+1)(2k+1)} \qquad \text{1}
\end{aligned}$$

It is true for $n = k + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(b)
$$\begin{aligned}
\sum_{r=30}^{120} \frac{1}{r(r-1)} &= \sum_{r=60}^{120} \frac{1}{r(r-1)} + \sum_{r=30}^{60} \frac{1}{r(r-1)} - \frac{1}{60(59)} \\
&= \frac{60+1}{120(60-1)} + \frac{30+1}{60(30-1)} - \frac{1}{3540} \qquad \text{1M} \\
&= \frac{91}{3480} \qquad \text{1A}
\end{aligned}$$

2. (a) When $n = 1$,

L.H.S. $= 1 \times 3^2 = 9$ and R.H.S. $= \frac{1}{6}(1)(1+1)[6(1)^2 + 14(1) + 7] = 9 =$ L.H.S.

It is true for $n = 1$. 1

Assume $1 \times 3^2 + 2 \times 5^2 + \ldots + k(2k+1)^2 = \frac{1}{6}k(k+1)(6k^2 + 14k + 7)$, where $k \in \mathbb{Z}^+$. 1

$$1 \times 3^2 + 2 \times 5^2 + 3 \times 7^2 + \ldots + k(2k+1)^2 + (k+1)[2(k+1)+1]^2$$

$$= \frac{1}{6}k(k+1)(6k^2 + 14k + 7) + (k+1)(2k+3)^2 \qquad 1$$

$$= \frac{1}{6}(k+1)[k(6k^2 + 14k + 7) + 6(2k+3)^2]$$

$$= \frac{1}{6}(k+1)[6k^3 + 38k^2 + 79k + 54]$$

$$= \frac{1}{6}(k+1)(k+2)(6k^2 + 26k + 27)$$

$$= \frac{1}{6}(k+1)[(k+1)+1][6(k+1)^2 + 14(k+1) + 7] \qquad 1$$

It is true for $n = k + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(b) (i) $1 \times 3^2 + 2 \times 5^2 + 3 \times 7^2 + \ldots + 20 \times 41^2$

$$= \sum_{r=1}^{20} r(2r+1)^2 \qquad \text{1M}$$

$$= \frac{1}{6}(20)(20+1)[6(20)^2 + 14(20) + 7] \qquad \text{1M}$$

$$= 188\,090 \qquad \text{1A}$$

(ii) $11 \times 23^2 + 12 \times 25^2 + 13 \times 27^2 + \ldots + 20 \times 41^2$

$$= \sum_{r=11}^{20} r(2r+1)^2$$

$$= \sum_{r=1}^{20} r(2r+1)^2 - \sum_{r=1}^{10} r(2r+1)^2 \qquad \text{1M}$$

$$= 188\,090 - \frac{1}{6}(10)(10+1)[6(10)^2 + 14(10) + 7] \qquad \text{1M}$$

$$= 174\,395 \qquad \text{1A}$$

3. (a) When $n = 1$,

L.H.S. $= (-1) = -1$ and R.H.S. $= \dfrac{(-1)(2+1)-1}{4} = -1 =$ L.H.S.

It is true for $n = 1$. 1

Assume $\displaystyle\sum_{k=1}^{p}(-1)^k k = \frac{(-1)^p(2p+1)-1}{4}$, where $p \in \mathbb{Z}^+$. 1

$$\begin{aligned}\sum_{k=1}^{p+1}(-1)^k k &= \sum_{k=1}^{p}(-1)^k k + (-1)^{p+1}(p+1) \\ &= \frac{(-1)^p(2p+1)-1}{4} + (-1)^{p+1}(p+1) && 1 \\ &= \frac{(-1)^{p+1}[-(2p+1)+4(p+1)]-1}{4} \\ &= \frac{(-1)^{p+1}(2p+3)-1}{4} && 1 \\ &= \text{R.H.S.}\end{aligned}$$

It is true for $n = p + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(b) $$\begin{aligned}\sum_{k=3}^{2017}(-1)^k(k+1) &= -\sum_{k=4}^{2018}(-1)^k k \\ &= -\sum_{k=1}^{2018}(-1)^k k + \sum_{k=1}^{3}(-1)^k k && \text{1M} \\ &= -\frac{(-1)^{2018}[2(2018)+1]-1}{4} + \frac{(-1)^3[2(3)+1]-1}{4} && \text{1M} \\ &= -1011 && \text{1A}\end{aligned}$$

4. (a) When $n = 1$,

L.H.S. $= (2-1)^2 = 1$ and R.H.S. $= \frac{1}{3}(1)(2-1)(2+1) = 1 =$ L.H.S.

It is true for $n = 1$. 1

Assume $\sum_{k=1}^{p}(2k-1)^2 = \frac{1}{3}p(2p-1)(2p+1)$, where $p \in \mathbb{Z}^+$. 1

$$\sum_{k=1}^{p+1}(2k-1)^2 = \sum_{k=1}^{p}(2k-1)^2 + (2p+1)^2$$

$$= \frac{1}{3}p(2p-1)(2p+1) + (2p+1)^2 \quad 1$$

$$= \frac{1}{3}(2p+1)[p(2p-1) + 3(2p+1)]$$

$$= \frac{1}{3}(2p+1)(2p^2+5p+3)$$

$$= \frac{1}{3}(p+1)(2p+1)(2p+3) \quad 1$$

It is true for $n = p+1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(b) $$\sum_{k=111}^{222}\left(k-\frac{1}{2}\right)^2 = \frac{1}{4}\sum_{k=1}^{222}(2k-1)^2 - \frac{1}{4}\sum_{k=1}^{110}(2k-1)^2 \quad 1\text{M}$$

$$= \frac{1}{4} \times \frac{1}{3}(222)(443)(445) - \frac{1}{4} \times \frac{1}{3}(110)(219)(221) \quad 1\text{M}$$

$$= 3\,203\,340 \quad 1\text{A}$$

5. (a) When $n = 1$,

L.H.S. $= 6 \times 10 = 60$ and R.H.S. $= \frac{2}{3}(1)(2 + 21 + 67) = 60 =$ L.H.S.

It is true for $n = 1$. 1

Assume $6 \times 10 + 8 \times 12 + \ldots + (2k + 4)(2k + 8) = \frac{2}{3}k(2k^2 + 21k + 67)$, where $k \in \mathbb{Z}^+$. 1M

$$
\begin{aligned}
&6 \times 10 + 8 \times 12 + \ldots + (2k + 4)(2k + 8) + (2k + 6)(2k + 10) \\
&= \frac{2}{3}k(2k^2 + 21k + 67) + (2k + 6)(2k + 10) && \text{1M} \\
&= \frac{2}{3}[(2k^3 + 21k^2 + 67k) + 3(2k + 6)(k + 5)] \\
&= \frac{2}{3}(2k^3 + 27k^2 + 115k + 90) \\
&= \frac{2}{3}(k + 1)(2k^2 + 25k + 90) && \text{1M} \\
&= \frac{2}{3}(k + 1)[2(k + 1)^2 + 21(k + 1) + 67]
\end{aligned}
$$

It is true for $n = k + 1$. 1

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(b)
$$
\begin{aligned}
\sum_{r=2}^{30}(r + 2)(r + 4) &= \frac{1}{4}\left[\sum_{r=1}^{30}(2r + 4)(2r + 8) - (6)(10)\right] && \text{1M} \\
&= \frac{1}{4}\left[\frac{2}{3}(30)[2(30)^2 + 21(30) + 67] - 60\right] && \text{1M} \\
&= 12\,470 && \text{1A}
\end{aligned}
$$

6. (a) When $n = 1$,

L.H.S. $= \sum_{k=1}^{1}(-1)^k(2k-1)^2 = (-1)(1)^2 = -1$

R.H.S. $= \dfrac{(-1)^1(2-1)(2+1)+1}{2} = -1 = $ L.H.S.

It is true for $n = 1$. 1

Assume $\sum_{k=1}^{p}(-1)^k(2k-1)^2 = \dfrac{(-1)^p(2p-1)(2p+1)+1}{2}$, where $p \in \mathbb{Z}^+$. 1

$$\sum_{k=1}^{p+1}(-1)^k(2k-1)^2$$
$$= \sum_{k=1}^{p}(-1)^k(2k-1)^2 + (-1)^{p+1}(2p+1)^2$$
$$= \frac{(-1)^p(2p-1)(2p+1)+1}{2} + (-1)^{p+1}(2p+1)^2 \quad 1$$
$$= \frac{(-1)^{p+1}(2p+1)[-(2p-1)+2(2p+1)]+1}{2}$$
$$= \frac{(-1)^{p+1}(2p+1)(2p+3)+1}{2}$$

It is true when $n = p + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(b) $\sum_{k=99}^{200}(-1)^{k+1}(2k-1)^2$

$$= -\sum_{k=1}^{200}(-1)^k(2k-1)^2 + \sum_{k=1}^{98}(-1)^k(2k-1)^2 \quad 1\text{M}$$
$$= -\frac{(-1)^{200}(400-1)(400+1)+1}{2} + \frac{(-1)^{98}(196-1)(196+1)+1}{2}$$
$$= -60\,792 \quad 1\text{A}$$

7. (a) When $n = 1$,

L.H.S. $= (3-2)(3-1) = 2$ and R.H.S. $= 1(3-1) = 2 =$ L.H.S.

It is true for $n = 1$. 1

Assume $\sum_{k=1}^{p}(3k-2)(3k-1) = p(3p^2-1)$, where $p \in \mathbb{Z}^+$. 1

$$\sum_{k=1}^{p+1}(3k-2)(3k-1) = \sum_{k=1}^{p}(3k-2)(3k-1) + (3p+1)(3p+2)$$

$$= p(3p^2-1) + (3p+1)(3p+2) \qquad 1$$

$$= 3p^3 + 9p^2 + 8p + 2$$

$$= (p+1)(3p^2+6p+2)$$

$$= (p+1)[3(p+1)^2-1] \qquad 1$$

It is true for $n = p+1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(b) (i) $2+5+8+\ldots+(3n-1) = \sum_{k=1}^{n}(3k-1)$

$$= 3\sum_{k=1}^{n} k - n$$

$$= 3 \times \frac{n(n+1)}{2} - n \qquad \text{1M}$$

$$= \frac{3n^2+n}{2} \qquad \text{1A}$$

(ii) $2^2+5^2+8^2+\ldots+(3n-1)^2 = \sum_{k=1}^{n}(3k-1)^2$

$$= \sum_{k=1}^{n}(3k-2)(3k-1) + \sum_{k=1}^{n}(3k-1)$$

$$= n(3n^2-1) + \frac{3n^2+n}{2} \qquad \text{1M}$$

$$= \frac{n(6n^2+3n-1)}{2} \qquad \text{1A}$$

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