M2REG-MAD-2324-ASM-SET 3-MATH
Suggested solutions

1. (a) $A^2 = \begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 28 & 8 \\ 24 & 12 \end{pmatrix}$

$A^2 - 2A = \begin{pmatrix} 28 & 8 \\ 24 & 12 \end{pmatrix} - 2\begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 18 & 6 \\ 18 & 6 \end{pmatrix}$ 1M

$6(A - 2I) = 6\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 6 \\ 18 & 6 \end{pmatrix}$

Thus, $A^2 - 2A = 6(A - 2I)$. 1

(b) When $n = 1$,

R.H.S. $= \frac{1}{4}(6 - 2)(A - 2I) + 2I = A =$ L.H.S.

It is true for $n = 1$. 1

Assume $A^k = \frac{1}{4}(6^k - 2^k)(A - 2I) + 2^k I$, where $k \in \mathbb{Z}$. 1

$$\begin{aligned} A^{k+1} &= A^k A \\ &= \left[\frac{1}{4}(6^k - 2^k)(A - 2I) + 2^k I\right] A \quad \text{1M} \\ &= \frac{1}{4}(6^k - 2^k)(A^2 - 2A) + 2^k A \\ &= \frac{1}{4}(6^k - 2^k)(6)(A - 2I) + 2^k A \\ &= \frac{1}{4}(6^{k+1} - 2^{k+1}(3))(A - 2I) + 2^k A \\ &= \frac{1}{4}(6^{k+1} - 2^{k+1})(A - 2I) - \frac{2}{4}(2^{k+1})(A - 2I) + 2^k A \\ &= \frac{1}{4}(6^{k+1} - 2^{k+1})(A - 2I) + 2^{k+1} I \end{aligned}$$

It is true for $n = k + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

2. (a) $A^2 = \begin{pmatrix} 4 & -7 \\ 3 & -5 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 1M+1A

Thus, the smallest value of n is 3. 1A

(b) (i) $A + A^2 + A^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 1A

(ii) $\sum_{k=1}^{335} A^k = (A + A^2 + A^3) + A^3(A + A^2 + A^3) + \ldots + A^{333}(A + A^2)$ 1M

$= A + A^2$

$= -A^3 = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ 1M+1A

3. (a) $A^2 = \begin{pmatrix} 5 & 16 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} 5 & 16 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 9 & 32 \\ -2 & -7 \end{pmatrix}$ 1A

$A^2 - A = \begin{pmatrix} 9 & 32 \\ -2 & -7 \end{pmatrix} - \begin{pmatrix} 5 & 16 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 16 \\ -1 & -4 \end{pmatrix}$ 1A

$A - I = \begin{pmatrix} 5 & 16 \\ -1 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 16 \\ -1 & -4 \end{pmatrix}$

Thus, $A^2 - A = A - I$. 1

(b) $A^2 - A = A - I$

$A^3 - A^2 = A(A^2 - A) = A - I$ 1A

$A^4 - A^3 = A(A^3 - A^2) = A - I$

$\vdots$

$A^n - A^{n-1} = A - I$

$(A^2 - A) + (A^3 - A^2) + \ldots + (A^n - A^{n-1}) = (n-1)(A - I)$ 1M

$A^n - A = (n-1)(A - I)$

$A^n = nA - (n-1)I$

$= \begin{pmatrix} 4n+1 & 16n \\ -n & 1-4n \end{pmatrix}$ 1A

4. (a) $A^2 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 0 & 4 & -3 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ 1A

$$-A^3 + A^2 + A - I = -\begin{pmatrix} 0 & 4 & -3 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 & -2 \\ 1 & -1 & 1 \\ 1 & -2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ 1

$$= \mathbf{0}$$

(b) (i) Note that $-A^{k+3} + A^{k+2} + A^{k+1} - A^k = A^k(-A^3 + A^2 + A - I) = \mathbf{0}$ for non-negative integers k.

$$\mathbf{0} = \sum_{k=0}^{n-3}(-A^{k+3} + A^{k+2} + A^{k+1} - A^k)$$ 1M

$$= \sum_{k=0}^{n-3}(-A^{k+3} + A^{k+2}) + \sum_{k=0}^{n-3}(A^{k+1} - A^k)$$

$$= (-A^n + A^2) + (A^{n-2} - I)$$ 1M

$$A^n = A^{n-2} + A^2 - I$$ 1

(ii) $A^{98} = A^{96} + A^2 - I$ 1M

$$= A^{94} + 2A^2 - 2I$$

$$= A^2 + 48A^2 - 48I$$ 1A

$$= 49\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} - 48\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 49 & -49 \\ 0 & 50 & -49 \\ 0 & 49 & -48 \end{pmatrix}$$ 1A

5. (a) (i) $\begin{pmatrix} a & -3 \\ -3 & a+8 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\begin{pmatrix} a+9 \\ -3a-27 \end{pmatrix} = \begin{pmatrix} \lambda \\ -3\lambda \end{pmatrix}$

We have $\lambda = a + 9$. 1A

(ii) $\begin{pmatrix} a & -3 \\ -3 & a+8 \end{pmatrix}\begin{pmatrix} b \\ 1 \end{pmatrix} = \mu \begin{pmatrix} b \\ 1 \end{pmatrix}$

$\begin{pmatrix} ab-3 \\ -3b+a+8 \end{pmatrix} = \begin{pmatrix} \mu b \\ \mu \end{pmatrix}$

We have $ab - 3 = \mu b$ and $-3b + a + 8 = \mu$.

$ab - 3 = (-3b + a + 8)b$ 1M

$3b^2 - 8b - 3 = 0$

$b = 3 \quad \text{or} \quad -\frac{1}{3}$ (rejected)

$\mu = -3(3) + a + 8 = a - 1$ 1A

(iii) (1) $M^T M = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$ 1A

(2) When $n = 1$,

R.H.S. $= \frac{1}{10} MDM^T$

$= \frac{1}{10}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} a+9 & 0 \\ 0 & a-1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix} a+9 & 3a-3 \\ -3a-27 & a-1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} a & -3 \\ -3 & a+8 \end{pmatrix}$

It is true for $n = 1$. 1

Assume $A^k = \frac{1}{10} MD^k M^T$, where $k \in \mathbb{Z}^+$. 1

$A^{k+1} = A^k A$

$= \frac{1}{10} MD^k M^T \left(\frac{1}{10} MDM^T\right)$ 1M

$= \frac{1}{100} MD^k (M^T M) DM^T$

$= \frac{1}{100} MD^k (10I) DM^T$

$= \frac{1}{10} MD^{k+1} M^T$

It is true for $n = k + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(iv) Put $a = 2$, then $\lambda = 11$, $\mu = 1$ and $A = \begin{pmatrix} 2 & -3 \\ -3 & 10 \end{pmatrix}$.

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -3 & 10 \end{pmatrix}^{2019} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

$$\begin{pmatrix} x & y \end{pmatrix} \left(\frac{1}{10} MDM^T \right) \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

$$\frac{1}{10} \begin{pmatrix} x - 3y & 3x + y \end{pmatrix} \begin{pmatrix} 11^{2019} & 0 \\ 0 & 1^{2019} \end{pmatrix} \begin{pmatrix} x - 3y \\ 3x + y \end{pmatrix} = \mathbf{0}$$ 1M

$$\begin{pmatrix} 11^{2019}(x - 3y) & 3x + y \end{pmatrix} \begin{pmatrix} x - 3y \\ 3x + y \end{pmatrix} = \mathbf{0}$$

$$\begin{pmatrix} 11^{2019}(x - 3y)^2 + (3x + y)^2 \end{pmatrix} = \mathbf{0}$$

We have $11^{2019}(x - 3y)^2 + (3x + y)^2 = 0$.

Thus, $x - 3y = 0$ and $3x + y = 0$. 1M

Solving, we have $x = y = 0$. 1

6. (a) (i) $(X+Y)^2 = (X+Y)(X+Y)$

$$= X^2 + XY + YX + Y^2$$

$$= X^2 + 2XY + Y^2 \qquad 1$$

(ii) $C_r^n + C_{r-1}^n = \dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)!}$

$$= \frac{n![(n+1-r)+r]}{r!(n+1-r)!} \qquad 1\text{M}$$

$$= \frac{(n+1)!}{r!(n+1-r)!}$$

$$= C_r^{n+1} \qquad 1$$

(iii) When $n = 1$, R.H.S. $= X + Y =$ L.H.S.

It is true when $n = 1$. 1

Assume $(X+Y)^k = \displaystyle\sum_{r=0}^{k} C_r^k X^{k-r} Y^r$, where $k \in \mathbb{Z}^+$. 1

$$(X+Y)^{k+1} = (X+Y)^k (X+Y)$$

$$= \left(\sum_{r=0}^{k} C_r^k X^{k-r} Y^r\right)(X+Y) \qquad 1$$

$$= (X^{k+1} + C_1^k X^k Y + C_2^k X^{k-1} Y^2 + \ldots + XY^k)$$
$$+ (X^k Y + C_1^k X^{k-1} Y^2 + C_2^k X^{k-2} Y^3 + \ldots + Y^{k+1})$$

$$= X^{k+1} + (C_1^k + C_0^k) X^k Y + (C_2^k + C_1^k) X^{k-1} Y^2 + \ldots + Y^{k+1}$$

$$= X^{k+1} + C_1^{k+1} X^k Y + C_2^{k+1} X^{k-1} Y^2 + \ldots + Y^{k+1} \qquad 1\text{M}$$

$$= \sum_{r=0}^{k+1} C_r^{k+1} X^{k+1-r} Y^r \qquad 1$$

It is true when $n = k + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(b) Let $X = I$ and $Y = \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ such that $XY = YX$ and $X + Y = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

$$Y^2 = \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y^3 = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0} \qquad 1\text{A}$$

$$\begin{aligned}
\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}^{100} &= (X+Y)^{100} \\
&= \sum_{r=0}^{100} C_r^{100} X^{100-r} Y^r \\
&= I + C_1^{100} Y + C_2^{100} Y^2 && \text{1M} \\
&= \begin{pmatrix} 1 & 200 & 30\,100 \\ 0 & 1 & 300 \\ 0 & 0 & 1 \end{pmatrix} && \text{1A}
\end{aligned}$$

7. (a) (i) $A = \dfrac{1}{b-a+1}(M - aI) = \dfrac{1}{b-a+1}\begin{pmatrix} b-a & b-a \\ 1 & 1 \end{pmatrix}$

$B = \dfrac{1}{b-a+1}(I + bI - M) = \dfrac{1}{b-a+1}\begin{pmatrix} 1 & a-b \\ -1 & b-a \end{pmatrix}$

$AB = \dfrac{1}{(b-a+1)^2}\begin{pmatrix} b-a & b-a \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & a-b \\ -1 & b-a \end{pmatrix}$ 1M

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 1A

$BA = \dfrac{1}{(b-a+1)^2}\begin{pmatrix} 1 & a-b \\ -1 & b-a \end{pmatrix}\begin{pmatrix} b-a & b-a \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A + B = \dfrac{1}{b-a+1}\left[\begin{pmatrix} b-a & b-a \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & a-b \\ -1 & b-a \end{pmatrix}\right]$

$= \dfrac{1}{b-a+1}\begin{pmatrix} b-a+1 & 0 \\ 0 & b-a+1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 1A

(ii) $A^2 = A(I - B)$ and $B^2 = B(I - A)$ 1M

$= A - AB$ $\quad$ $= B - BA$

$= A$ $\quad$ $= B$ 1

(iii) When $n = 1$,

R.H.S. $= (b+1)A + aB = \dfrac{b+1}{b-a+1}(M - aI) + \dfrac{a}{b-a+1}(I + bI - M) = M =$ L.H.S.

It is true for $n = 1$. 1

Assume $M^k = (b+1)^k A + a^k B$, where $k \in \mathbb{Z}^+$. 1

$M^{k+1} = M^k M$

$= [(b+1)^k A + a^k B]M[(b+1)A + aB]$ 1M

$= (b+1)^{k+1}A + (b+1)^k aAB + a^k(b+1)BA + a^{k+1}B$

$= (b+1)^{k+1}A + a^{k+1}B$

It is true for $n = k + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(b) Put $b = -1$ and $a = -1$ such that $M = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$. 1M

$$\begin{aligned} \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}^{999} &= (-1)^{999} \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}^{999} && \text{1M} \\ &= (-1)[(-2)^{999}A + (-1)^{999}B] && \text{1M} \\ &= 2^{999}A + B \\ &= \begin{pmatrix} 2^{1000} - 1 & 2^{1000} - 2 \\ -2^{999} + 1 & -2^{999} + 2 \end{pmatrix} && \text{1A} \end{aligned}$$

8. (a) $A^2 = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 17 & -8 \\ 16 & -7 \end{pmatrix}$

$A^2 - mA + nI = \begin{pmatrix} 17 - 5m + n & -8 + 2m \\ 16 - 4m & -7 + m + n \end{pmatrix}$

We have $17 - 5m + n = 0$, $-8 + 2m = 0$, $16 - 4m = 0$ and $-7 + m + n = 0$. 1M

Solving, we have $m = 4$ and $n = 3$. 1A+1A

(b) (i) $\alpha = 3$ and $\beta = 1$ 1A

$P^2 = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -8 & 8 \end{pmatrix}$

$Q^2 = \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ 8 & -4 \end{pmatrix}$ 1A

$PQ = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$QP = \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 1A

(ii) Note that $PQ = QP = \mathbf{0}$, $P^2 = -2P$ and $Q^2 = 2Q$.

$$\begin{aligned} -(A - 3I) + 3(A - I) &= -P + 3Q \\ A &= -\frac{1}{2}P + \frac{3}{2}Q && \text{1A} \end{aligned}$$

$$\begin{aligned} A^{100} &= \left(-\frac{1}{2}P + \frac{3}{2}Q\right)^{100} \\ &= \left(\frac{-1}{2}P\right)^{100} + C_1^{100}\left(-\frac{1}{2}P\right)^{99}\left(\frac{3}{2}Q\right) + C_2^{100}\left(-\frac{1}{2}P\right)^{98}\left(\frac{3}{2}\right)^2 + \ldots + \left(\frac{3}{2}Q\right)^{100} && \text{1M} \\ &= \frac{1}{2^{100}}P^{100} + \frac{3^{100}}{2^{100}}Q^{100} \\ &= \frac{1}{2^{100}}[(-2)^{99}]P + \frac{3^{100}}{2^{100}}(2^{99})Q && \text{1A} \\ &= -\frac{1}{2}P + \frac{3^{100}}{2}Q \\ &= \begin{pmatrix} -1 + 2 \times 3^{100} & 1 - 3^{100} \\ -2 + 2 \times 3^{100} & 2 - 3^{100} \end{pmatrix} && \text{1A} \end{aligned}$$

9. (a) $\begin{vmatrix} a & b \\ 1 & c \end{vmatrix} = 0$

$ac - b = 0$

$b = ac$ 1A

Thus, $A = \begin{pmatrix} a & ac \\ 1 & c \end{pmatrix}$.

$A^2 = \begin{pmatrix} a & ac \\ 1 & c \end{pmatrix}\begin{pmatrix} a & ac \\ 1 & c \end{pmatrix}$

$= \begin{pmatrix} a^2 + ac & a^2c + ac^2 \\ a + c & ac + c^2 \end{pmatrix}$

$= (a+c)\begin{pmatrix} a & ac \\ 1 & c \end{pmatrix}$

$= (a+c)A$ 1M

$A^3 = A^2 A$

$= (a+c)A^2$

$= (a+c)^2 A$

Thus, $A^{n+1} = (a+c)^n A$. 1

(b) (i) $\Delta = (a+c)^2 - 4(\det A)$ 1M

$= (a+c)^2 - 4(ac - b)$

$= (a-c)^2 + 4b$

$\geq 4b$

> 0

Thus, α and β are distinct real numbers. 1

(ii) $A^2 = \begin{pmatrix} a & b \\ 1 & c \end{pmatrix}\begin{pmatrix} a & b \\ 1 & c \end{pmatrix} = \begin{pmatrix} a^2 + b & ab + bc \\ a + c & b + c^2 \end{pmatrix}$

$(\alpha + \beta)A - \alpha\beta I = (a+c)A - (ac - b)I$

$= (a+c)\begin{pmatrix} a & b \\ 1 & c \end{pmatrix} - (ac - b)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} a^2 + b & ac + bc \\ a + c & b + c^2 \end{pmatrix}$

$= A^2$ 1

(iii) (1) $XY = (A - \alpha I)(A - \beta I)$

$= A^2 - (\alpha + \beta)A + \alpha\beta I$

$= \mathbf{0}$

$YX = (A - \beta I)(A - \alpha I)$

$= A^2 - (\alpha + \beta)A + \alpha\beta I$

$= \mathbf{0}$ 1

$$\det X = (a-\alpha)(c-\alpha) - b$$
$$= \alpha^2 - (a+c)\alpha + ac - b$$
$$= \alpha^2 - (a+c)\alpha + \det A$$
$$\det Y = (a-\beta)(c-\beta) - b$$
$$= \beta^2 - (a+c)\beta + ac - b$$
$$= \beta^2 - (a+c)\beta + \det A$$

Since α and β are roots of the equation $x^2 - (a+c)x + \det A = 0$,

$\det X = \det Y = 0$. 1

(2) $\lambda X + \mu Y = \lambda(A - \alpha I) + \mu(A - \beta I)$
$$= (\lambda+\mu)A - (\alpha\lambda + \beta\mu)I$$

We have $\lambda + \mu = 1$ and $\alpha\lambda + \beta\mu = 0$. 1M

$\alpha\lambda + \beta(1-\lambda) = 0$ and $\alpha(1-\mu) + \beta\mu = 0$

$(\alpha - \beta)\lambda = -\beta$ $\quad$ $(-\alpha+\beta)\mu = -\alpha$

$\lambda = \dfrac{\beta}{\beta - \alpha}$ $\quad$ $\mu = \dfrac{\alpha}{\alpha - \beta}$ 1

(c) Put $a = 4$, $b = 3$ and $c = 2$, we have $\det A = 5$ and $b = 3 > 0$.

Let α and β be the roots of $x^2 - (4+2)x + 5 = 0$, we have $\alpha = 1$ and $\beta = 5$. 1A

Then $\lambda = \dfrac{5}{5-1} = \dfrac{5}{4}$ and $\mu = \dfrac{1}{1-5} = -\dfrac{1}{4}$.

$$\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}^n = \left[\frac{5}{4}\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\right]^n$$ 1M

$$= \left(\frac{5}{4}\right)^n \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}^n + \left(-\frac{1}{4}\right)^n \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}^n$$

$$= \left(\frac{5}{4}\right)^n (3+1)^{n-1} \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} + \left(-\frac{1}{4}\right)^n (-1-3)^{n-1} \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$$ 1M

$$= \frac{5^n}{4}\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{3(5^n)+1}{4} & \dfrac{3(5^n)-3}{4} \\ \dfrac{5^n-1}{4} & \dfrac{5^n+3}{4} \end{pmatrix}$$ 1A

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