

M2REG-DEFINT-2223-ASM-SET 6-MATH**Suggested solutions**

1. (a) Let $u = a - x$. Then $du = -dx$. 1M

$$\begin{aligned}\int_0^a f(a-x) dx &= - \int_a^0 f(u) du \\ &= \int_0^a f(u) du && \text{1M} \\ &= \int_0^a f(x) dx && \text{1}\end{aligned}$$

We have $\int_0^a f(x) dx = \int_0^a f(a-x) dx$.

$$\begin{aligned}\int_0^a f(x) dx &= \frac{1}{2} \left[\int_0^a f(x) dx + \int_0^a f(a-x) dx \right] \\ &= \frac{1}{2} \int_0^a [f(x) + f(a-x)] dx && \text{1}\end{aligned}$$

(b) $\int_0^3 \frac{dx}{e^{3-2x} + 1}$

$$\begin{aligned}&= \frac{1}{2} \int_0^3 \left[\frac{1}{e^{3-2x} + 1} + \frac{1}{e^{3-2(3-x)} + 1} \right] dx && \text{1M} \\ &= \frac{1}{2} \int_0^3 \left[\frac{1}{e^{3-2x} + 1} + \frac{1}{e^{2x-3} + 1} \times \frac{e^{3-2x}}{e^{3-2x}} \right] dx \\ &= \frac{1}{2} \int_0^3 \frac{1 + e^{3-2x}}{e^{3-2x} + 1} dx && \text{1M} \\ &= \frac{1}{2} \left[x \right]_0^3 \\ &= \frac{1}{2} 3 && \text{1A}\end{aligned}$$

$$\begin{aligned}
2. \quad (a) \quad (i) \quad \cos \theta (2 \cos 4\theta - 2 \cos 2\theta + 1) &= 2 \cos \theta \cos 4\theta - 2 \cos \theta \cos 2\theta + \cos \theta \\
&= (\cos 5\theta + \cos 3\theta) - (\cos 3\theta + \cos \theta) + \cos \theta \\
&= \cos 5\theta
\end{aligned}$$

1M

$$\text{Thus, } \frac{\cos 5\theta}{\cos \theta} = 2 \cos 4\theta - 2 \cos 2\theta + 1.$$

1

$$\begin{aligned}
(ii) \quad \frac{\cos 5\left(\frac{\pi}{4} - t\right)}{\cos\left(\frac{\pi}{4} - t\right)} &= 2 \cos\left(\frac{\pi}{4} - t\right) - 2 \cos 2\left(\frac{\pi}{4} - t\right) + 1 \\
\frac{\cos \frac{5\pi}{4} \cos 5t + \sin \frac{5\pi}{4} \sin 5t}{\cos \frac{\pi}{4} \cos t + \sin \frac{\pi}{4} \sin t} &= -2 \cos 4t - 2 \sin 2t + 1 \\
\frac{-\frac{\sqrt{2}}{2} \cos 5t - \frac{\sqrt{2}}{2} \sin 5t}{\frac{\sqrt{2}}{2} \cos t + \frac{\sqrt{2}}{2} \sin t} &= -2 \cos 4t - 2 \sin 2t + 1 \\
\frac{\cos 5t + \sin 5t}{\cos t + \sin t} &= 2 \cos 4t + 2 \sin 2t - 1
\end{aligned}$$

1M

1

$$(b) \quad (i) \quad \text{Let } u = a - x. \text{ Then } du = -dx.$$

1M

$$\begin{aligned}
\int_0^a f(a-x) dx &= -\int_a^0 f(u) du \\
&= \int_0^a f(u) du \\
&= \int_0^a f(x) dx
\end{aligned}$$

1M

1

$$\begin{aligned}
(ii) \quad \int_0^{\frac{\pi}{2}} \frac{\cos 5x}{\cos x + \sin x} dx &= \int_0^{\frac{\pi}{2}} \frac{\cos 5\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin 5x}{\cos x + \sin x} dx
\end{aligned}$$

1M

1

$$\begin{aligned}
(c) \quad \int_0^{\frac{\pi}{2}} \frac{\cos 5x}{\cos x + \sin x} dx &= \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\cos 5x}{\cos x + \sin x} dx + \int_0^{\frac{\pi}{2}} \frac{\sin 5x}{\cos x + \sin x} dx \right] \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} (2 \cos 4x + 2 \sin 2x - 1) dx \\
&= \frac{1}{2} \left[\frac{\sin 4x}{2} - \cos 2x - x \right]_0^{\frac{\pi}{2}} \\
&= 1 - \frac{\pi}{4}
\end{aligned}$$

1M

1M

1A

3. (a) Let $u = a - x$. Then $du = -dx$. 1M

$$\begin{aligned}\int_0^a f(a-x) dx &= -\int_a^0 f(u) du \\ &= \int_0^a f(u) du && \text{1M} \\ &= \int_0^a f(x) dx && \text{1}\end{aligned}$$

(b) $\int_0^{\frac{\pi}{4}} \ln(\cos x) dx = \int_0^{\frac{\pi}{4}} \ln \left[\cos \left(\frac{\pi}{4} - x \right) \right] dx$ 1M

$$= \int_0^{\frac{\pi}{4}} \ln \left(\cos \frac{\pi}{4} \cos x + \sin \frac{\pi}{4} \sin x \right) dx$$
 1M

$$= \int_0^{\frac{\pi}{4}} \ln \frac{\cos x + \sin x}{\sqrt{2}} dx$$
 1

(c) $\int_0^{\frac{\pi}{4}} \ln(\cos x) dx = \int_0^{\frac{\pi}{4}} \ln \frac{\cos x + \sin x}{\sqrt{2}} dx$

$$= \int_0^{\frac{\pi}{4}} \ln(\cos x + \sin x) dx - \int_0^{\frac{\pi}{4}} \sqrt{2} dx$$
 1M

$$\int_0^{\frac{\pi}{4}} \ln \frac{\cos x}{\cos x + \sin x} dx = -\ln \sqrt{2} \left[x \right]_0^{\frac{\pi}{4}}$$
 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{1 + \tan x} dx = -\frac{\pi \ln 2}{8}$$
 1

(d) Note that $\frac{d}{dx} [\ln(1 + \tan x)] = \frac{\sec^2 x}{1 + \tan x}$.

$$\int_0^{\frac{\pi}{4}} \frac{x \sec^2 x}{1 + \tan x} dx = \left[x \ln(1 + \tan x) \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1 + \tan x) dx$$
 1M

$$= \frac{\pi \ln 2}{4} + \int_0^{\frac{\pi}{4}} \ln \frac{1}{1 + \tan x} dx$$

$$= \frac{\pi \ln 2}{4} + \left(-\frac{\pi \ln 2}{8} \right)$$
 1M

$$= \frac{\pi \ln 2}{8}$$
 1A

4. (a) Let $u = \pi - x$. Then $du = -dx$. 1M

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\pi} e^{\frac{1}{1+\sin x}} f(x) dx &= - \int_{\frac{\pi}{2}}^0 e^{\frac{1}{1+\sin(\pi-u)}} f(\pi-u) du \\ &= \int_0^{\frac{\pi}{2}} e^{\frac{1}{1+\sin u}} [-f(u)] du \end{aligned}$$

1M

$$= - \int_0^{\frac{\pi}{2}} e^{\frac{1}{1+\sin x}} f(x) dx$$

1

$$\begin{aligned} \int_0^{\pi} e^{\frac{1}{1+\sin x}} f(x) dx &= \int_0^{\frac{\pi}{2}} e^{\frac{1}{1+\sin x}} f(x) dx + \int_{\frac{\pi}{2}}^{\pi} e^{\frac{1}{1+\sin x}} f(x) dx \\ &= \int_0^{\frac{\pi}{2}} e^{\frac{1}{1+\sin x}} f(x) dx - \int_0^{\frac{\pi}{2}} e^{\frac{1}{1+\sin x}} f(x) dx \\ &= 0 \end{aligned}$$

1

(b) Let $v = \pi - x$. Then $dv = -dx$. 1M

$$\begin{aligned} \int_0^{\pi} \sin x g(x) dx &= \int_0^{\frac{\pi}{2}} \sin x g(x) dx + \int_{\frac{\pi}{2}}^{\pi} \sin x g(x) dx \\ &= \int_0^{\frac{\pi}{2}} \sin x g(x) dx - \int_{\frac{\pi}{2}}^0 \sin(\pi-u) g(\pi-u) du \\ &= \int_0^{\frac{\pi}{2}} \sin x g(x) dx + \int_0^{\frac{\pi}{2}} \sin u g(u) du \\ &= \int_0^{\frac{\pi}{2}} \sin x g(x) dx + \int_0^{\frac{\pi}{2}} \sin x g(x) dx \\ &= 2 \int_0^{\frac{\pi}{2}} \sin x g(x) dx \end{aligned}$$

1M

1

(c) (i) Put $f(x) = \cos x \sin x$.

$$f(\pi - x) = \cos(\pi - x) \sin(\pi - x) = -\cos x \sin x = -f(x)$$

1M

$$\int_0^{\pi} e^{\frac{1}{1+\sin x}} \cos x \sin x dx = 0$$

1A

Let $w = \cos x$. Then $dw = -\sin x dx$. 1M

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin x \cos^2 x dx &= - \int_1^0 u^2 du \\ &= -\frac{1}{3} \left[u^3 \right]_1^0 \\ &= \frac{1}{3} \end{aligned}$$

1A

1A

$$\begin{aligned} \text{(ii)} \quad \int_0^{\pi} \sin 2x (\cos x + e^{\frac{1}{1+\sin x}}) dx &= \int_0^{\pi} (\sin 2x \cos x + e^{\frac{1}{1+\sin x}} \sin 2x) dx \\ &= 2 \int_0^{\pi} \sin x \cos^2 x dx + 2 \int_0^{\pi} e^{\frac{1}{1+\sin x}} \sin x \cos x dx \\ &= 2 \int_0^{\pi} \sin x \cos^2 x dx \end{aligned}$$

1M

Put $g(x) = \cos^2 x$. $g(\pi - x) = \cos^2(\pi - x) = \cos^2 x = g(x)$. 1M

$$\begin{aligned} \int_0^\pi \sin 2x(\cos x + e^{\frac{1}{1+\sin x}}) dx &= 2 \int_0^\pi \sin x \cos^2 x dx \\ &= 4 \int_0^{\frac{\pi}{2}} \sin x \cos^2 x dx \\ &= 4 \left(\frac{1}{3} \right) \\ &= \frac{4}{3} \end{aligned} \quad \text{1A}$$

5. (a) Let $u = x^3$. Then $du = 3x^2 dx$. 1M

$$\begin{aligned} \int x^2 \sin x^3 dx &= \frac{1}{3} \int \sin u du \\ &= -\frac{\cos u}{3} + \text{constant} \\ &= -\frac{\cos x^3}{3} + \text{constant} \end{aligned} \quad \text{1A}$$

(i) Let $u = a + b - x$. Then $du = -dx$. 1M

$$\begin{aligned} &\int_a^b \frac{f(x)}{f(a+b-x) + f(x)} dx \\ &= - \int_b^a \frac{f(a+b-u)}{f(u) + f(a+b-u)} du \\ &= \int_a^b \frac{f(a+b-u)}{f(u) + f(a+b-u)} du \quad \text{1M} \\ &= \int_a^b \frac{f(a+b-x)}{f(x) + f(a+b-x)} dx \quad \text{1A} \\ &\int_a^b \frac{f(x)}{f(a+b-x) + f(x)} dx \\ &= \frac{1}{2} \left[\int_a^b \frac{f(x)}{f(a+b-x) + f(x)} dx + \int_a^b \frac{f(a+b-x)}{f(a+b-x) + f(x)} dx \right] \quad \text{1M} \\ &= \frac{1}{2} \int_a^b dx \quad \text{1A} \\ &= \frac{1}{2} \left[x \right]_a^b \\ &= \frac{b-a}{2} \quad \text{1} \end{aligned}$$

(ii) Let $u = x^3$. Then $du = 3x^2 dx$. 1M

$$\begin{aligned} &\int_{\sqrt[3]{\log 5}}^{\sqrt[3]{\log 6}} \frac{x^2 \sin x^3}{\sin(\log 30 - x^3) + \sin x^3} dx \\ &= \frac{1}{3} \int_{\log 5}^{\log 6} \frac{\sin u}{\sin(\log 5 + \log 6 - u) + \sin u} du \quad \text{1M} \\ &= \frac{1}{3} \cdot \frac{\log 6 - \log 5}{2} \quad \text{1M} \\ &= \frac{1}{6} \log \frac{6}{5} \quad \text{1A} \end{aligned}$$

6. (a) Let $u = \pi - x$. Then $du = -dx$. 1M

$$\begin{aligned}
 \int_0^\pi x f(\sin x) dx &= - \int_\pi^0 (\pi - u) f(\sin(\pi - u)) du \\
 &= \int_0^\pi (\pi - u) f(\sin u) du & 1M \\
 &= \pi \int_0^\pi f(\sin u) du - \int_0^\pi u f(\sin u) du \\
 &= \pi \int_0^\pi f(\sin x) dx - \int_0^\pi x f(\sin x) dx \\
 2 \int_0^\pi x f(\sin x) dx &= \pi \int_0^\pi f(\sin x) dx \\
 \int_0^\pi x f(\sin x) dx &= \frac{\pi}{2} \int_0^\pi f(\sin x) dx & 1
 \end{aligned}$$

(b) (i) Let $v = -x$. Then $dv = -dx$. 1M

$$\begin{aligned}
 \int_{-\pi}^0 \frac{g(x)}{1 + e^x} dx &= - \int_\pi^0 \frac{g(-v)}{1 + e^{-v}} dv \\
 &= \int_0^\pi \frac{e^v g(v)}{e^v (1 + e^{-v})} dv \\
 &= \int_0^\pi \frac{e^v g(v)}{1 + e^v} dv \\
 &= \int_0^\pi \frac{e^x g(x)}{1 + e^x} dx & 1
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \int_{-\pi}^\pi \frac{g(x)}{1 + e^x} dx &= \int_{-\pi}^0 \frac{g(x)}{1 + e^x} dx + \int_0^\pi \frac{g(x)}{1 + e^x} dx \\
 &= \int_0^\pi \frac{e^x g(x)}{1 + e^x} dx + \int_0^\pi \frac{g(x)}{1 + e^x} dx \\
 &= \int_0^\pi \frac{(1 + e^x)g(x)}{1 + e^x} dx \\
 &= \int_0^\pi g(x) dx & 1
 \end{aligned}$$

(c) Put $g(x) = \frac{\sqrt{2 + \cos x}}{\sqrt{2 + \cos x} + \sqrt{2 - \cos x}}$.

$$g(-x) = \frac{\sqrt{2 + \cos(-x)}}{\sqrt{2 + \cos(-x)} + \sqrt{2 - \cos(-x)}} = \frac{\sqrt{2 + \cos x}}{\sqrt{2 + \cos x} + \sqrt{2 - \cos x}} = g(x)$$

$g(x)$ is a continuous even function on $[-\pi, \pi]$.

$$\begin{aligned} & \int_{-\pi}^{\pi} \frac{\sqrt{2 + \cos x}}{(1 + e^x)(\sqrt{2 + \cos x} + \sqrt{2 - \cos x})} dx \\ &= \int_0^{\pi} g(x) dx \\ &= \left[xg(x) \right]_0^{\pi} - \int_0^{\pi} xg'(x) dx \end{aligned}$$

1M

$$g'(x) = \frac{\frac{-\sin x}{2\sqrt{2+\cos x}}(\sqrt{2+\cos x} + \sqrt{2-\cos x}) - (\sqrt{2+\cos x})\left(\frac{-\sin x}{2\sqrt{2+\cos x}} + \frac{\sin x}{2\sqrt{2-\cos x}}\right)}{(\sqrt{2+\cos x} + \sqrt{2-\cos x})^2}$$

1M

$$= \frac{-\sin x}{2\sqrt{3 + \sin^2 x} + (3 + \sin^2 x)}$$

1A

Take $f(x) = \frac{-x}{2\sqrt{3+x^2} + 3+x^2}$ such that it is continuous on $[0, 1]$ and $g'(x) = f(\sin x)$.

$$\begin{aligned} & \int_{-\pi}^{\pi} \frac{\sqrt{2 + \cos x}}{(1 + e^x)(\sqrt{2 + \cos x} + \sqrt{2 - \cos x})} dx \\ &= \left[xg(x) \right]_0^{\pi} - \int_0^{\pi} xf(\sin x) dx \\ &= \pi g(\pi) - \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx \\ &= \pi g(\pi) - \frac{\pi}{2} \int_0^{\pi} g'(x) dx \\ &= \pi g(\pi) - \frac{\pi}{2} \left[g(x) \right]_0^{\pi} \\ &= \frac{\pi}{2} \left(\frac{\sqrt{2-1}}{\sqrt{2-1} + \sqrt{2-1}} + \frac{\sqrt{2+1}}{\sqrt{2+1} + \sqrt{2-1}} \right) \\ &= \frac{\pi}{2} \end{aligned}$$

1M

1A

7. (a) Let $u = a + b - x$. Then $du = -dx$.

$$\int_a^b f(a + b - x) dx = - \int_b^a f(u) du \quad 1M$$

$$= \int_a^b f(u) du \quad 1M$$

$$= \int_a^b f(x) dx \quad 1$$

(b) $\int_0^{\frac{\pi}{4}} \ln \sin 2x dx = \int_0^{\frac{\pi}{4}} \ln \sin \left[2 \left(\frac{\pi}{4} - x \right) \right] dx \quad 1M$

$$= \int_0^{\frac{\pi}{4}} \ln \cos 2x dx \quad 1$$

(c) Let $2x = t$. Then $2 dx = dt$. 1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \ln \sin 4x dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln \sin 2t dt \\ &= \frac{1}{2} \left[\int_0^{\frac{\pi}{4}} \ln \sin 2t dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \sin 2t dt \right] \quad 1M \end{aligned}$$

$$= \frac{1}{2} \left[\int_0^{\frac{\pi}{4}} \ln \sin 2t dt + \int_0^{\frac{\pi}{4}} \ln \sin \left[2 \left(u + \frac{\pi}{4} \right) \right] du \right] \quad \text{where } u = t - \frac{\pi}{4} \quad 1M$$

$$\begin{aligned} &= \frac{1}{2} \left[\int_0^{\frac{\pi}{4}} \ln \sin 2t dt + \int_0^{\frac{\pi}{4}} \ln \cos 2u du \right] \\ &= \frac{1}{2} \left[\int_0^{\frac{\pi}{4}} \ln \sin 2x dx + \int_0^{\frac{\pi}{4}} \ln \sin 2x dx \right] \\ &= \int_0^{\frac{\pi}{4}} \ln \sin 2x dx \quad 1 \end{aligned}$$

(d) $\int_0^{\frac{\pi}{4}} \ln \sin 4x dx = \int_0^{\frac{\pi}{4}} (\ln 2 + \ln \sin 2x + \ln \cos 2x) dx$

$$= \ln 2 \left[x \right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \ln \sin 2x dx + \int_0^{\frac{\pi}{4}} \ln \cos 2x dx \quad 1M$$

$$= \frac{\pi \ln 2}{4} + 2 \int_0^{\frac{\pi}{4}} \ln \sin 4x dx \quad 1M$$

$$\int_0^{\frac{\pi}{4}} \ln \sin 4x dx = -\frac{\pi \ln 2}{4} \quad 1A$$