M2REG-AOI-2223-ASM-SET 4-MATH
Suggested solutions

1. (a) $\displaystyle\int_0^k \frac{1}{\sqrt{4-x}}\,\mathrm{d}x = \int_k^3 \frac{1}{\sqrt{4-x}}\,\mathrm{d}x$ 1M+1M

$$\left[-2\sqrt{4-x}\right]_0^k = \left[-2\sqrt{4-x}\right]_k^3$$

$$4-2\sqrt{4-k} = 2\sqrt{4-k}-2$$

$$\sqrt{4-k} = \frac{3}{2}$$

$$k = \frac{7}{4}$$ 1A

(b) Required volume

$$= \pi\int_0^{\frac{7}{4}} \frac{1}{4-x}\,\mathrm{d}x$$ 1M

$$= \pi\left[-\ln|4-x|\right]_0^{\frac{7}{4}}$$ 1M

$$= \pi\ln\frac{16}{9}$$ 1A

2. (a) $y = x-3$ 1A

(b) Required area $\displaystyle = \int_2^3 \left[\left(x-3+\frac{16}{x^2-4x+7}\right)-(x-3)\right]\mathrm{d}x$ 1M

$$= \int_2^3 \frac{16}{x^2-4x+7}\,\mathrm{d}x$$

$$= \int_2^3 \frac{16}{(x-2)^2+3}\,\mathrm{d}x$$ 1M

Let $x-2=\sqrt{3}\tan\theta$. Then $\mathrm{d}x = \sqrt{3}\sec^2\theta\,\mathrm{d}\theta$

When $x=2$, $\theta=0$; when $x=3$, $\theta=\dfrac{\pi}{6}$. 1M

Required area $\displaystyle = \int_0^{\frac{\pi}{6}} \frac{16\sqrt{3}\sec^2\theta}{(\sqrt{3}\tan\theta)^2+3}\,\mathrm{d}\theta$

$$= \frac{16\sqrt{3}}{3}\int_0^{\frac{\pi}{6}} \frac{\sec^2\theta}{\sec^2\theta}\,\mathrm{d}\theta$$ 1M

$$= \frac{16\sqrt{3}}{3}\int_0^{\frac{\pi}{6}} \mathrm{d}\theta$$

$$= \frac{16\sqrt{3}}{3}\Big[\theta\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{8\sqrt{3}\pi}{9}$$ 1A

3. (a) Let $w = x + 1$. Then $\mathrm{d}w = \mathrm{d}x$. 1M

$$\text{Area} = \int_0^u \frac{x}{\sqrt{x+1}}\,\mathrm{d}x$$

$$= \int_1^{u+1} \frac{w-1}{\sqrt{w}}\,\mathrm{d}w \qquad \text{1A}$$

$$= \int_1^{u+1} \left(w^{\frac{1}{2}} - w^{-\frac{1}{2}}\right)\mathrm{d}w$$

$$= \left[\frac{2}{3}w^{\frac{3}{2}} - 2w^{\frac{1}{2}}\right]_1^{u+1} \qquad \text{1M}$$

$$= \frac{2}{3}(u-2)\sqrt{u+1} + \frac{4}{3} \qquad \text{1A}$$

(b) Let $A = \frac{2}{3}(u-2)\sqrt{u+1} + \frac{4}{3}$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}$$

$$= \frac{2}{3}\left[(u-2)\cdot\frac{1}{2\sqrt{u+1}} + \sqrt{u+1}\right]\frac{\mathrm{d}u}{\mathrm{d}t} \qquad \text{1M+1A}$$

$$= \frac{u}{\sqrt{u+1}}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}$$

When $OQ = 8$,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{8}{\sqrt{8+1}}\cdot 3$$

$$= 8$$

Required rate is 8 square units per second. 1A

4. (a) Let $x = 3 + 3\sin\theta$. Then $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$.

$$\int \sqrt{x(6-x)}\,\mathrm{d}x = \int \sqrt{(3+3\sin\theta)(3-3\sin\theta)}3\cos\theta\,\mathrm{d}\theta \qquad \text{1M}$$

$$= 9\int \cos^2\theta\,\mathrm{d}\theta$$

$$= \frac{9}{2}\int (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{9\theta}{2} + \frac{9\sin 2\theta}{4} + \text{constant} \qquad \text{1A}$$

$$= \frac{9}{2}\sin^{-1}\frac{x-3}{3} + \frac{9}{2}\sin\theta\cos\theta + \text{constant}$$

$$= \frac{9}{2}\sin^{-1}\frac{x-3}{3} + \frac{9}{2}\left(\frac{x-3}{3}\right)\sqrt{1-\left(\frac{x-3}{3}\right)^2} + \text{constant} \qquad \text{1M}$$

$$= \frac{9}{2}\sin^{-1}\frac{x-3}{3} + \frac{x-3}{2}\sqrt{6x-x^2} + \text{constant} \qquad \text{1A}$$

(b) In the first quadrant, $y^2 = \sqrt{6x-x^2} = \sqrt{x(6-x)}$. 1M

$$\text{Volume} = \pi\int_0^6 \sqrt{x(6-x)}\,\mathrm{d}x \qquad \text{1M}$$

$$= \pi\left[\frac{9}{2}\sin^{-1}\frac{x-3}{3} + \frac{x-3}{2}\sqrt{6x-x^2}\right]_0^6 \qquad \text{1M}$$

$$= \frac{9\pi^2}{2} \qquad \text{1A}$$

5. (a) Volume $= \pi \int_4^{4+h} y \, dy$ 1M

$= \pi \left[\frac{y^2}{2} \right]_4^{4+h}$

$= \frac{\pi}{2}(h^2 + 8h)$ 1

(b) (i) $V = V_1 + \frac{\pi}{2}(h^2 + 8h)$, where V_1 is the volume of the frustum and $h = H - 4$ for $4 \le H \le 12$.

$\frac{dV}{dt} = \frac{\pi(2h+8)}{2}\frac{dh}{dt}$ 1A

$8 = \frac{\pi[2(7-4)+8]}{2}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{8}{7\pi}$

Required rate is $\frac{8}{7\pi}$ cm/s. 1A

(ii) Consider the cone that makes up the frustum. Let the height of the cone be H_0 cm.

$\frac{H_0 - 4}{H_0} = \frac{2}{4}$ 1M

$H_0 = 8$ 1A

$V = \frac{\pi}{3}(4^2)(8)\left[1 - \left(\frac{8-H}{8}\right)^3\right]$ 1M+1A

$= \frac{128\pi}{3}\left[1 - \left(1 - \frac{H}{8}\right)^3\right]$ 1

(iii) Volume of the upper part $= \frac{\pi}{2}[8^2 + 8(8)] = 64\pi$ cm^3.

Volume of milk leaked $= \frac{\pi}{2} \times 130 = 65\pi$ cm$^3 > 64\pi$ cm^3.

Depth of remaining milk is less than 4 cm.

$64\pi + \frac{128\pi}{3}\left[1 - \left(1 - \frac{4}{8}\right)^3\right] - 65\pi = \frac{128\pi}{3}\left[1 - \left(1 - \frac{H}{8}\right)^3\right]$ 1M

$\left(1 - \frac{H}{8}\right)^3 = \frac{19}{128}$

$1 - \frac{H}{8} = \left(\frac{19}{128}\right)^{\frac{1}{3}}$ 1A

$V = \frac{128\pi}{3}\left[1 - \left(1 - \frac{H}{8}\right)^3\right]$

$\frac{dV}{dt} = \frac{128\pi}{3}\left[0 - 3\left(1 - \frac{H}{8}\right)^2\left(-\frac{1}{8}\right)\frac{dH}{dt}\right]$

$= 16\pi\left(1 - \frac{H}{8}\right)^2\frac{dH}{dt}$ 1A

$-\frac{\pi}{2} = 16\pi\left(\frac{19}{128}\right)^{\frac{2}{3}}\frac{dH}{dt}$

$\frac{dH}{dt} \approx -0.111$

Required rate is 0.111 cm/s. 1A

6. (a) (i) Put $y = 0$ into $3x^2 - 2y - 4 = 0$, we have $x = \pm\sqrt{\frac{4}{3}}$. 1A

The points $\left(\pm\sqrt{\frac{4}{3}},\ 0\right)$ lies on the curve $3x^2 + 2y + k = 0$,

$$3 \times \frac{4}{3} + 2(0) + k = 0$$

$$k = -4$$ 1A

(ii) The equation of BC can be rewritten as $y = -\frac{3}{2}x^2 + 2$.

The coordinates of vertex are $(0,\ 2)$. 1A

(b) (i) When $0 \le h \le 2$,

$$\text{Volume} = \pi \int_0^h \left[\frac{2y+4}{3} - \frac{4-2y}{3}\right] \mathrm{d}x$$ 1M+1A

$$= \pi \int_0^h \frac{4y}{3}\,\mathrm{d}y$$

$$= \pi \left[\frac{2y^2}{3}\right]_0^h$$ 1M

$$= \frac{2\pi h^2}{3} \text{ cubic units}$$ 1A

(ii) When $2 < h \le 6$,

$$\text{Volume} = \frac{2\pi(2)^2}{3} + \pi \int_2^h \frac{2y+4}{3}\,\mathrm{d}y$$ 1M

$$= \frac{8\pi}{3} + \pi\left[\frac{4y}{3} + \frac{y^2}{3}\right]_2^h$$

$$= \pi\left(\frac{h^2}{3} + \frac{4h}{3} - \frac{4}{3}\right) \text{ cubic units}$$ 1A

(c) Let V cubic units be the volume of water in the cup at time t seconds.

When $V = \frac{28\pi}{3}$, we have $V > \frac{8\pi}{3}$ and so $h > 2$. 1M

$$\frac{28\pi}{3} = \pi\left(\frac{h^2}{3} + \frac{4h}{3} - \frac{4}{3}\right)$$

$$h^2 + 4h - 32 = 0$$ 1M

$$h = 4 \quad \text{or} \quad -8 \text{ (rejected)}$$

For $2 < h \le 6$, $\frac{\mathrm{d}V}{\mathrm{d}t} = \pi\left(\frac{2h}{3} + \frac{4}{3}\right)\frac{\mathrm{d}h}{\mathrm{d}t}$ 1M

When $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{8\pi}{3}$ and $h = 4$,

$$\frac{8\pi}{3} = \pi\left(\frac{2\times 4}{3} + \frac{4}{3}\right)\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{2}{3}$$

Required rate is $\frac{2}{3}$ units per second. 1A

7. (a) $\sqrt{(x+4)^2+y^2}+\sqrt{(x-4)^2+y^2}=10$

$$\sqrt{x^2+y^2+8x+16}=10-\sqrt{x^2+y^2-8x+16} \quad \text{1M}$$

$$x^2+y^2+8x+16=100+x^2+y^2-8x+16-20\sqrt{x^2+y^2-8x+16}$$

$$5\sqrt{x^2+y^2-8x+16}=25-4x \quad \text{1M}$$

$$25(x^2+y^2-8x+16)=625-200x+16x^2$$

$$9x^2+25y^2=225$$

$$\frac{x^2}{25}+\frac{y^2}{9}=1 \quad \text{1}$$

(b) (i) Let V cubic units be the volume of water.

$$V=\pi\int_{-3}^{-3+h}25\left(1-\frac{y^2}{9}\right)\mathrm{d}y \quad \text{1M}$$

$$=\frac{25\pi}{9}\int_{-3}^{-3+h}(9-y^2)\,\mathrm{d}y$$

$$=\frac{25\pi}{9}\left[9y-\frac{y^3}{3}\right]_{-3}^{-3+h} \quad \text{1A}$$

$$=\frac{25\pi}{27}(9h^2-h^3) \quad \text{1A}$$

(ii) $$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{25\pi}{27}(18h-3h^2)\frac{\mathrm{d}h}{\mathrm{d}t} \quad \text{1M}$$

$$30-5=\frac{25\pi}{9}(6h-h^2)\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{9}{\pi(6h-h^2)} \quad \text{1A}$$

$$=\frac{9}{-\pi(h-3)^2+9\pi} \quad \text{1M}$$

Since $0<h<6$, $\frac{\mathrm{d}h}{\mathrm{d}t}$ is minimum when the denominator is maximum.

Required depth of water is 3 units. 1A

(iii) $$V=\int\frac{-\pi}{800}(t+100)\,\mathrm{d}t \quad \text{1M}$$

$$=-\frac{\pi t^2}{1600}-\frac{\pi t}{8}+C$$

When $t=0$, $h=3$ and $V=\frac{25\pi}{27}[9(3)^2-3^3]=50\pi$.

So, $C=50\pi$. 1A

When $V=0$,

$$-\frac{\pi t^2}{1600}-\frac{\pi t}{8}+50\pi=0 \quad \text{1M}$$

$$t^2+200t-80\,000=0$$

$$t=200 \quad \text{or} \quad -400 \text{ (rejected)}$$

The time required is 200 minutes. 1A

8. (a) $f(-2) = (-2)^3 + p(-2)^2 + q(-2) + 1 = 9$ 1M

$$4p - 2q = 16$$

$\dfrac{dy}{dx} = 3x^2 + 2px + q$ 1A

$3(-2)^2 + 2p(-2) + q = 0$

$$4p - q = 12$$

Solving, we have $p = 2$ and $q = -4$. 1A

(b) $\dfrac{d^2y}{dx^2} = 6x + 4$. So, $\left.\dfrac{d^2y}{dx^2}\right|_{x=-2} = 6(-2) + 4 = -8 < 0$ 1M

So, P is a maximum point and is not a minimum point. 1A

(c) $\dfrac{dy}{dx} = 3x^2 + 4x - 4 = 0$ 1M

$$x = \frac{2}{3} \quad \text{or} \quad -2$$

When $x = \dfrac{2}{3}$, $y = -\dfrac{13}{27}$ and $\dfrac{d^2y}{dx^2} = 8 > 0$.

So, maximum point is $(-2,\ 9)$ and minimum point is $\left(\dfrac{2}{3},\ -\dfrac{13}{27}\right)$. 1A

(d) When $\dfrac{d^2y}{dx^2} = 6x + 4 = 0$, $x = -\dfrac{2}{3}$.

x	$x < -\frac{2}{3}$	$x > -\frac{2}{3}$
$\frac{d^2y}{dx^2}$	$-$	$+$

1M

The point of inflexion is $\left(-\dfrac{2}{3},\ \dfrac{115}{27}\right)$. 1A

(e) The equation of L is $y = 9$.

$$x^3 + 2x^2 - 4x + 1 = 9$$ 1M

$$x^3 + 2x^2 - 4x - 8 = 0$$

$$(x+2)^2(x-2) = 0$$

$$x = \pm 2$$

Area $= \displaystyle\int_{-2}^{2} [9 - (x^3 + 2x^2 - 4x + 1)]\, dx$ 1M

$= \left[-\dfrac{x^4}{4} - \dfrac{2x^3}{3} + 2x^2 + 8x \right]_{-2}^{2}$ 1M

$= \dfrac{64}{3}$ 1A

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