M2REG-AOI-2223-ASM-SET 3-MATH
Suggested solutions

1. Required volume

$$= \pi \int_1^2 [(e^y - 1) - (-2)]^2 \, \mathrm{d}y - \pi \int_1^2 [(2y - 1) - (-2)]^2 \, \mathrm{d}y \qquad \text{1M+1M}$$

$$= \pi \int_1^2 [(e^y + 1)^2 - (2y + 1)^2] \, \mathrm{d}y$$

$$= \pi \int_1^2 (e^{2y} + 2e^y - 4y^2 - 4y) \, \mathrm{d}y$$

$$= \pi \left[\frac{e^{2y}}{2} + 2e^y - \frac{4y^3}{3} - 2y^2 \right]_1^2 \qquad \text{1M}$$

$$= \pi \left(\frac{e^4}{2} + \frac{3e^2}{2} - 2e - \frac{46}{3} \right) \qquad \text{1A}$$

2. Required volume

$$= \pi \int_1^4 \left(2 - \frac{\sqrt{y}}{2} \right)^2 \mathrm{d}y - \pi \int_1^4 \left(2 - \sqrt{y} \right)^2 \mathrm{d}y \qquad \text{1M+1M}$$

$$= \pi \int_1^4 \left(2\sqrt{y} - \frac{3y}{4} \right) \mathrm{d}y$$

$$= \pi \left[\frac{4y^{\frac{3}{2}}}{3} - \frac{3y^2}{8} \right]_1^4 \qquad \text{1M}$$

$$= \frac{89\pi}{24} \qquad \text{1A}$$

3. Volume $= \pi \int_{-3}^3 \frac{9 - y^2}{2} \, \mathrm{d}y - \pi \int_{-2}^2 \frac{4 - y^2}{2} \, \mathrm{d}y$ 2M+1A

$$= \pi \left[\frac{9y}{2} - \frac{y^3}{6} \right]_{-3}^3 - \pi \left[2y - \frac{y^3}{6} \right]_{-2}^2 \qquad \text{1M}$$

$$= \frac{38\pi}{3} \qquad \text{1A}$$

4. (a) $y = 4y - y^2$ 1M

$y^2 - 3y = 0$

$y = 0 \quad \text{or} \quad 3$

The coordinates of P are $\left(\sqrt{3},\ 3\right)$. 1A

(b) Required volume

$= \pi \int_0^3 (4y - y^2)\,\mathrm{d}y - \pi \int_0^3 y\,\mathrm{d}y$ 1M+1M

$= \pi \int_0^3 (3y - y^2)\,\mathrm{d}y$

$= \pi \left[\frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3$ 1M

$= \frac{9\pi}{2}$ 1A

5. (a) Solve $\begin{cases} y = -3x - 3 \\ y = 6 \end{cases}$, the coordinates of A are $(-3,\ 6)$. 1A

Solve $\begin{cases} y = x + 3 \\ y = -3x - 3 \end{cases}$, the coordinates of B are $\left(-\frac{3}{2},\ \frac{3}{2}\right)$. 1A

Solve $\begin{cases} y = x + 3 \\ y = 6 \end{cases}$, the coordinates of C are $(3,\ 6)$. 1A

(b) Required volume

$= \pi \int_{-3}^{-\frac{3}{2}} [8 - (-3x - 3)]^2\,\mathrm{d}x + \pi \int_{-\frac{3}{2}}^{3} [8 - (x + 3)]^2\,\mathrm{d}x - \pi(8 - 6)^2(3 + 3)$ 1M+1M

$= \pi \int_{-3}^{-\frac{3}{2}} (9x^2 + 66x + 121)\,\mathrm{d}x + \pi \int_{-\frac{3}{2}}^{3} (x^2 - 10x + 25)\,\mathrm{d}x - 24\pi$

$= \pi \left[3x^3 + 33x^2 + 121x \right]_{-3}^{-\frac{3}{2}} + \pi \left[\frac{x^3}{3} - 5x^2 + 25x \right]_{-\frac{3}{2}}^{3} - 24\pi$ 1M

$= \frac{189\pi}{2}$ 1A

6. (a) $\frac{dy}{dx} = (1-x)^{-\frac{1}{2}}(-1)$ 1M

$= -\frac{1}{\sqrt{1-x}}$

At $A\ (0,\ 2)$, $\frac{dy}{dx} = -\frac{1}{\sqrt{1-0}} = -1$. Slope of $L = \frac{-1}{-1} = 1$. 1M

The equation of L is $y = x + 2$. 1A

(b) L: $x = y - 2$ and C: $x = 1 - \frac{y^2}{4}$.

Required volume $= \pi \int_0^2 [1 - (y-2)]^2\, dy - \pi \int_0^2 \left[1 - \left(1 - \frac{y^2}{4}\right)\right]^2 dy$ 1M+1A

$= \pi \int_0^2 (9 - 6y + y^2)\, dy - \pi \int_0^2 \frac{y^4}{16}\, dy$

$= \pi \left[\frac{y^3}{3} - 3y^2 + 9y\right]_0^2 - \pi \left[\frac{y^5}{80}\right]_0^2$ 1M

$= \frac{124\pi}{15}$ 1A

7. (a) $6 - x^2 = 4 - x$ 1M

$0 = x^2 - x - 2$

$x = 2 \quad \text{or} \quad -1$

When $x = 2$, $y = 4 - 2 = 2$; when $x = -1$, $y = 4 - (-1) = 5$.

The coordinates of A and B are $(2,\ 2)$ and $(-1,\ 5)$ respectively. 1A+1A

(b) Volume $= \pi \int_{-1}^2 [(6 - x^2) - 2]^2\, dx - \pi \int_{-1}^2 [(4 - x) - 2]^2\, dx$ 1M+1M

$= \pi \int_{-1}^2 (x^4 - 8x^2 + 16)\, dx - \pi \int_{-1}^2 (4 - 4x + x^2)\, dx$

$= \pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_{-1}^2 - \pi \left[4x - 2x^2 + \frac{x^3}{3}\right]_{-1}^2$ 1M

$= \frac{108\pi}{5}$ 1A

8. (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(e^x \cos x) = e^x \cos x - e^x \sin x$ 1A

(b) $\displaystyle\int e^x \cos 2x \, \mathrm{d}x = e^x \cos 2x + 2\int e^x \sin 2x \, \mathrm{d}x$ 1M

$\displaystyle = e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x \, \mathrm{d}x$ 1M

$\displaystyle 5\int e^x \cos 2x \, \mathrm{d}x = e^x \cos 2x + 2e^x \sin 2x + \text{constant}$

$\displaystyle \int e^x \cos 2x \, \mathrm{d}x = \frac{e^x \cos 2x}{5} + \frac{2e^x \sin 2x}{5} + \text{constant}$ 1A

(c) $e^x(\sin 2x + 1) = e^x(\cos 2x + 1)$

$\sin 2x = \cos 2x$

$\tan 2x = 1$

$x = \dfrac{\pi}{8}$ or $\dfrac{5\pi}{8}$ 1A

$\displaystyle \text{Volume} = \pi \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} [e^{2x}(\sin 2x + 1)^2 - e^{2x}(\cos 2x + 1)^2] \, \mathrm{d}x$ 1M+1A

$\displaystyle = \pi \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} e^{2x}(\sin^2 2x - \cos^2 2x) \, \mathrm{d}x + 2\pi \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} e^{2x}(\sin 2x - \cos 2x) \, \mathrm{d}x$

$\displaystyle = -\pi \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} e^{2x} \cos 4x \, \mathrm{d}x + 2\pi \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} e^{2x}(\sin 2x - \cos 2x) \, \mathrm{d}x$

$\displaystyle = \frac{\pi}{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} e^u \cos 2u \, \mathrm{d}u + \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} e^u(\sin u - \cos u) \, \mathrm{d}u$ where $u = 2x$ 1M

$\displaystyle = \frac{\pi}{2}\left[\frac{e^u \cos 2u}{5} + \frac{2e^u \sin 2u}{5}\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - \pi\left[e^u \cos u\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$ 1M

$\displaystyle = \frac{\pi}{5}\left(e^{\frac{\pi}{4}} - e^{\frac{5\pi}{4}}\right) + \frac{\pi}{\sqrt{2}}\left(e^{\frac{5\pi}{4}} + e^{\frac{\pi}{4}}\right)$ 1A

9. (a) $1 - \cos 4\theta + 2\cos 2\theta \sin^2 2\theta$

$= 1 - (1 - 2\sin^2 2\theta) + 2(2\cos^2\theta - 1)\sin^2 2\theta$ 1M

$= 4\cos^2\theta \sin^2 2\theta$ 1

(b) $\displaystyle\int_0^{n\pi} \cos^2\theta \sin^2 2\theta \,\mathrm{d}\theta = \frac{1}{4}\int_0^{n\pi} (1 - \cos 4\theta + 2\cos 2\theta \sin^2 2\theta)\,\mathrm{d}\theta$ 1M

$\displaystyle = \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{n\pi} + \frac{1}{4}\int_0^{n\pi} \sin^2 2\theta \,\mathrm{d}\sin 2\theta$ 1M

$\displaystyle = \frac{n\pi}{4} + \frac{1}{4}\left[\frac{\sin^3 2\theta}{3}\right]_0^{n\pi}$

$\displaystyle = \frac{n\pi}{4}$ 1

(c) (i) Required area

$\displaystyle = \int_0^{\pi} \cos\theta \sin 2\theta \,\mathrm{d}\theta$ 1M

$\displaystyle = \frac{1}{2}\int_0^{\pi} (\sin x + \sin 3x)\,\mathrm{d}x$ 1M

$\displaystyle = \frac{1}{2}\left[-\cos x - \frac{\cos 3x}{3}\right]_0^{\pi}$

$\displaystyle = \frac{4}{3}$ 1A

(ii) Required volume

$\displaystyle = \pi\int_0^{\pi} \cos^2 \sin^2 2\theta \,\mathrm{d}\theta$ 1M

$\displaystyle = \pi \times \frac{\pi}{4}$ 1M

$\displaystyle = \frac{\pi^2}{4}$ 1A

10. (a) (i) The coordinates of A and B are $\left(a^2,\ a\right)$ and $\left(b^2,\ -b\right)$ respectively.

Required equation is

$$y-a=\frac{a+b}{a^2-b^2}(x-a^2) \qquad \text{1M}$$

$$y-a=\frac{1}{a-b}(x-a^2)$$

$$x-(a-b)y-ab=0 \qquad \text{1A}$$

(ii) Required area

$$=\int_{-b}^{a}[(a-b)y+ab-y^2]\,\mathrm{d}y \qquad \text{1M}$$

$$=\left[\frac{(a-b)y^2}{2}+aby-\frac{y^3}{3}\right]_{-b}^{a} \qquad \text{1M}$$

$$=\frac{1}{6}\left[[3(a-b)a^2+6a^2b-2a^3]-[3(a-b)b^2-6ab^2+2b^3]\right]$$

$$=\frac{1}{6}(a^3+3a^2b+3ab^2+b^3)$$

$$=\frac{1}{6}(a+b)^3 \qquad \text{1}$$

(b) (i) $4-(a-b)(0)-ab=0$

$$ab=4 \qquad \text{1}$$

(ii) Let A be the area of the shaded region.

$$A=\frac{1}{6}(a+b)^3$$

$$=\frac{1}{6}\left(a+\frac{4}{a}\right)^3$$

$$\frac{\mathrm{d}A}{\mathrm{d}a}=\frac{1}{2}\left(a+\frac{4}{a}\right)^2\left(1-\frac{4}{a^2}\right) \qquad \text{1M}$$

When $\dfrac{\mathrm{d}A}{\mathrm{d}a}=0$,

$$1-\frac{4}{a^2}=0$$

$$a^2=4$$

$$a=2 \quad \text{or} \quad -2 \text{ (rejected)}$$

a	$0<a<2$	$a>2$
$\dfrac{\mathrm{d}A}{\mathrm{d}a}$	$-$	$+$

1M

The area of the shaded region is a minimum when $a=2$. 1A

(c) When $a=2$, we have $b=2$ and the equation of AB is $x=4$.

Required volume

$$= \pi \int_{-2}^{2} [4 - (-1)]^2 \, dy - \pi \int_{-2}^{2} [y^2 - (-1)^2] \, dy$$ 1M

$$= 25\pi \int_{-2}^{2} dy - \pi \int_{-2}^{2} (y^4 + 2y^2 + 1) \, dy$$

$$= 25\pi \Big[y \Big]_{-2}^{2} - \pi \left[\frac{y^5}{5} + \frac{2y^3}{3} + y \right]_{-2}^{2}$$ 1M

$$= \frac{1088\pi}{15}$$ 1A

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