

M2REG-AOI-2223-ASM-SET 2-MATH

Suggested solutions

1. (a) Put $x = 2y - 2$ into $y^2 = mx - 3$,

$$y^2 = m(2y - 2) - 3 \quad 1M$$

$$y^2 - 2my + (2m + 3) = 0$$

$$\Delta = (2m)^2 - 4(1)(2m + 3) = 0 \quad 1M$$

$$4m^2 - 8m - 12 = 0$$

$$m = 3 \quad \text{or} \quad -1 \text{ (rejected)} \quad 1A$$

- (b) $y^2 - 2(3)y + [2(3) + 3] = 0$

$$y^2 - 6y + 9 = 0$$

$$y = 3$$

When $y = 3$, $x = 2(3) - 2 = 4$. The coordinates of P are $(4, 3)$. 1A

(c) Area = $\int_{-2}^4 \frac{x+2}{2} dx - \int_1^4 \sqrt{3x-3} dx$ 1M+1M

$$= \left[\frac{x^2}{4} + x \right]_{-2}^4 - \frac{1}{3} \left[\frac{2}{3} (3x-3)^{\frac{3}{2}} \right]_1^4 \quad 1M$$

$$= 3 \quad 1A$$

2. Required area = $-\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos y dy$ 1M

$$= -\left[\sin y \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \quad 1M$$

$$= 2 \quad 1A$$

3. (a) $\int \ln y dy = y \ln y - \int dy$ 1M

$$= y \ln y - y + \text{constant} \quad 1A$$

(b) Required area = $-\int_1^{e^2} (-\ln y) dy$ 1M+1M

$$= \left[y \ln y - y \right]_1^{e^2} \quad 1M$$

$$= e^2 + 1 \quad 1A$$

4. (a) $y^2 = 4x - 12$

$$2y \frac{dy}{dx} = 4 \quad 1M$$

$$\frac{dy}{dx} = \frac{2}{y}$$

Slope of tangent at $P = \frac{2}{-2} = -1$

Required equation is

$$y + 2 = -1(x - 4) \quad 1M$$

$$y = -x + 2 \quad 1A$$

(b) Area = $\int_{-2}^0 \left[\frac{y^2 + 12}{4} - (-y + 2) \right] dy \quad 1M$

$$= \int_{-2}^0 \left[\frac{y^2}{4} + y + 1 \right] dy$$

$$= \left[\frac{y^3}{12} + \frac{y^2}{2} + y \right]_{-2}^0 \quad 1M$$

$$= \frac{2}{3} \quad 1A$$

5. The coordinates of A , B and C are $(4, 0)$, $(0, 2)$ and $(9, 5)$ respectively. 1A

$$\text{Area} = \int_0^2 [(4 + y) - (4 - 2y)] dy + \int_2^5 [(4 + y) - (y^2 - 4y + 4)] dy \quad 1M+1M$$

$$= \left[\frac{3y^2}{2} \right]_0^2 + \left[\frac{-y^3}{3} + \frac{5y^2}{2} \right]_2^5 \quad 1M$$

$$= \frac{39}{2} \quad 1A$$

6. (a) $\int u(5^u) du = \frac{u(5^u)}{\ln 5} - \frac{1}{\ln 5} \int 5^u du \quad 1M$

$$= \frac{u(5^u)}{\ln 5} - \frac{5^u}{(\ln 5)^2} + C \quad 1A$$

(b) Required area = $\int_0^1 x(5^{2x}) dx \quad 1M$

Let $u = 2x$. Then $du = 2 dx$.

$$\text{Required area} = \frac{1}{4} \int_0^2 u(5^u) du \quad 1M$$

$$= \frac{1}{4} \left[\frac{u(5^u)}{\ln 5} - \frac{5^u}{(\ln 5)^2} \right]_0^2 \quad 1M$$

$$= \frac{1}{4} \left[\left(\frac{2(25)}{\ln 5} - \frac{25}{(\ln 5)^2} \right) - \left(0 - \frac{1}{(\ln 5)^2} \right) \right]$$

$$= \frac{25}{2 \ln 5} - \frac{6}{(\ln 5)^2} \quad 1A$$

7. (a) $\frac{dy}{dx} = \frac{1}{x+2}$ 1A

Slope of $L = \frac{1}{h+2}$

Equation of L is

$$y - \ln(h+2) = \frac{1}{h+2}(x - h)$$

$$y = \frac{x-h}{h+2} + \ln(h+2)$$
 1A

$$A = \int_0^h \left(\frac{x-h}{h+2} + \ln(h+2) - \ln(x+2) \right) dx$$
 1M

$$= \left[\frac{x^2 - 2hx}{2(h+2)} + \ln(h+2)x \right]_0^h - \int_0^h \ln(x+2) dx$$

$$= \frac{-h^2}{2h+4} + h \ln(h+2) - \left[x \ln(x+2) \right]_0^h + \int_0^h \frac{x}{x+2} dx$$
 1M

$$= \frac{-h^2}{2h+4} + h \ln(h+2) - h \ln(h+2) + \int_0^h \left(1 - \frac{2}{x+2} \right) dx$$

$$= \frac{-h^2}{2h+4} + \left[x - 2 \ln|x+2| \right]_0^h$$

$$= \frac{h^2 + 4h}{2h+4} - 2 \ln(h+2) + 2 \ln 2$$
 1

(b) $A = \frac{h}{2} + 1 - \frac{2}{h+2} - 2 \ln(h+2) + 2 \ln 2$

$$\frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt}$$
 1M

$$= \left(\frac{1}{2} + \frac{2}{(h+2)^2} - \frac{2}{h+2} \right) \cdot 3^{-t} (-\ln 3)$$
 1A

When $t = 1$, $h = 3^{-1} = \frac{1}{3}$,

$$\frac{dA}{dt} = \left(\frac{1}{2} + \frac{2}{\left(\frac{1}{3} + 2\right)^2} - \frac{2}{\frac{1}{3} + 2} \right) \cdot \frac{1}{3} \cdot (-\ln 3)$$

$$= -\frac{\ln 3}{294}$$

Required rate is $-\frac{\ln 3}{294}$ square units per second. 1A

$$8. \quad (a) \quad \int (\ln x)^2 dx = x(\ln x)^2 - 2 \int \ln x dx \quad 1M$$

$$= x(\ln x)^2 - 2x \ln x + 2 \int dx \quad 1M$$

$$= x(\ln x)^2 - 2x \ln x + 2x + \text{constant} \quad 1A$$

$$(b) \quad \text{Volume} = \pi \int_0^1 \left(\sqrt{x} \ln(x^2 + 1) \right)^2 dx \quad 1M$$

$$= \pi \int_0^1 x \left(\ln(x^2 + 1) \right)^2 dx$$

Let $u = x^2 + 1$. Then $du = 2x dx$.

$$\text{Volume} = \frac{\pi}{2} \int_1^2 (\ln u)^2 du \quad 1M$$

$$= \frac{\pi}{2} \left[u(\ln u)^2 - 2u \ln u + 2u \right]_1^2 \quad 1M$$

$$= \pi((\ln 2)^2 - 2 \ln 2 + 1) \quad 1A$$

$$9. \quad (a) \quad \text{The } x\text{-coordinate of } A \text{ is } 1. \quad 1A$$

$$x^2 - 1 = \frac{12}{x^2}$$

$$x^4 - x^2 - 12 = 0$$

$$x^2 = 4 \quad \text{or} \quad -3 \text{ (rejected)}$$

$$x = 2 \quad \text{or} \quad -2$$

The x -coordinate of B is 2. 1A

(b) Required area

$$= - \int_0^1 (x^2 - 1) dx + \int_1^2 (x^2 - 1) dx + \int_2^3 \frac{12}{x^2} dx \quad 1M+1A$$

$$= - \left[\frac{x^3}{3} - x \right]_0^1 + \left[\frac{x^3}{3} - x \right]_1^2 + \left[-\frac{12}{x} \right]_2^3 \quad 1M$$

$$= 4 \quad 1A$$

(c) Required volume

$$= \pi \int_0^2 (x^2 - 1)^2 dx + \pi \int_2^3 \left(\frac{12}{x^2} \right)^2 dx \quad 1M+1A$$

$$= \pi \int_0^2 (x^4 - 2x^2 + 1) dx + \pi \int_2^3 \frac{144}{x^4} dx$$

$$= \pi \left[\frac{x^5}{5} - \frac{2x^3}{3} + x \right]_0^2 + \pi \left[-\frac{48}{x^3} \right]_2^3 \quad 1M$$

$$= \frac{328\pi}{45} \quad 1A$$

10. (a) $g'(x) = e^x \cos x - e^x \sin x$ 1M

When $g'(x) = 0$,

$$e^x (\cos x - \sin x) = 0$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}$$

x	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \pi$
$g'(x)$	+	-

1M

Thus, G attains its maximum value only at $x = \frac{\pi}{4}$, and G has only one maximum point.

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(b) $\int e^{2x} \cos 2x \, dx = \frac{1}{2} e^{2x} \cos 2x + \int e^{2x} \sin 2x \, dx$ 1M

$$= \frac{1}{2} e^{2x} \cos 2x + \frac{1}{2} e^{2x} \sin 2x - \int e^{2x} \cos 2x \, dx$$
 1M

$$2 \int e^{2x} \cos 2x \, dx = \frac{1}{2} e^{2x} \cos 2x + \frac{1}{2} e^{2x} \sin 2x + \text{constant}$$

$$\int e^{2x} \cos 2x \, dx = \frac{1}{4} e^{2x} \cos 2x + \frac{1}{4} e^{2x} \sin 2x + \text{constant}$$
 1A

(c) Volume $= \pi \int_0^{\frac{\pi}{4}} e^{2x} \cos^2 x \, dx$ 1M

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{4}} e^{2x} (\cos 2x + 1) \, dx$$

$$= \frac{\pi}{2} \left[\frac{1}{4} e^{2x} \cos 2x + \frac{1}{4} e^{2x} \sin 2x \right]_0^{\frac{\pi}{4}} + \frac{\pi}{2} \left[\frac{1}{2} e^{2x} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{3\pi}{8} (e^{\frac{\pi}{2}} - 1)$$
 1A

11. (a) $\int (\ln x)^2 \, dx = x(\ln x)^2 - 2 \int \ln x \, dx$ 1M

$$= x(\ln x)^2 - 2x \ln x + 2 \int 1 \, dx$$
 1M

$$= x(\ln x)^2 - 2x \ln x + 2x + \text{constant}$$
 1A

(b) Required volume $= \pi \int_1^{e^3} (\ln y)^2 \, dy$ 1M

$$= \pi \left[y(\ln y)^2 - 2y \ln y + 2y \right]_1^{e^3}$$
 1M

$$= \pi(5e^3 - 2)$$
 1A

12. (a) (i) $f'(x) = \frac{162(x+1)^3 - 162(x-1)(3)(x+1)^2}{(x+1)^6}$
 $= \frac{324(2-x)}{(x+1)^4}$ 1A

$f''(x) = \frac{-324(x+1)^4 - 324(2-x)(4)(x+1)^3}{(x+1)^8}$
 $= \frac{972(x-3)}{(x+1)^5}$ 1A

(ii) When $f'(x) = 0$, $x = 2$.

x	$x < -1$	$-1 < x < 2$	$x > 2$
$f'(x)$	+	+	-

 1M

The maximum point is (2, 6). 1A

No minimum point.

When $f''(x) = 0$, $x = 3$.

x	$x < -1$	$-1 < x < 3$	$x > 3$
$f''(x)$	+	-	+

 1M

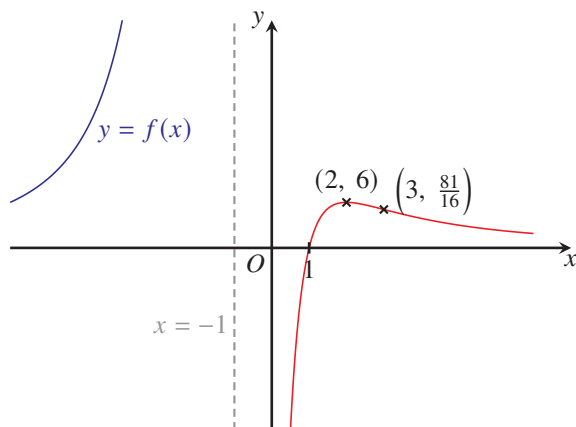
The point of inflexion is $\left(3, \frac{81}{16}\right)$. 1A

(b) Vertical asymptote is $x = -1$. 1A

Horizontal asymptote is $y = 0$. 1A

(c) (Shape, extrema and point of inflexion) 1A

(All correct) 1A



(d) Let $u = x + 1$. Then $du = dx$.

Volume $= \pi \int_0^1 \left[\frac{162(x-1)}{(x+1)^3} \right]^2 dx$ 1M

$= 26\,244\pi \int_1^2 \frac{(u-2)^2}{u^6} du$
 $= 26\,244\pi \int_1^2 (u^{-4} - 4u^{-5} + 4u^{-6}) du$ 1M

$= 26\,244 \left[\frac{u^{-3}}{-3} + u^{-4} - \frac{4u^{-5}}{5} \right]_1^2$
 $= \frac{67\,797\pi}{20}$ 1A