

M2REG-DEFINT-2223-ASM-SET 5-MATH

Suggested solutions

1. (a) Let $x = \tan \theta$. Then $dx = \sec^2 \theta d\theta$. 1M

$$\begin{aligned} \int_0^1 \frac{dx}{1+x^2} &= \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1+\tan^2 \theta} d\theta & 1A \\ &= \int_0^{\frac{\pi}{4}} d\theta \\ &= \left[\theta \right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{4} & 1A \end{aligned}$$

- (b) Let $u = \frac{\pi}{2} - x$. Then $du = -dx$. 1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sin x f(\sin 2x) dx &= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \sin \left(\frac{\pi}{2} - u \right) f \left[\sin 2 \left(\frac{\pi}{2} - u \right) \right] (-1) du \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos u f(\sin 2u) du & 1M \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x f(\sin 2x) dx & 1 \end{aligned}$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos x f(\sin 2x) dx &= \int_0^{\frac{\pi}{4}} \cos x f(\sin 2x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x f(\sin 2x) dx \\ &= \int_0^{\frac{\pi}{4}} \cos x f(\sin 2x) dx + \int_0^{\frac{\pi}{4}} \sin x f(\sin 2x) dx \\ &= \int_0^{\frac{\pi}{4}} (\cos x + \sin x) f(\sin 2x) dx & 1 \end{aligned}$$

$$(c) \int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} dx = \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{2 - \sin 2x} dx \quad 1M$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{1 + (1 - \sin 2x)} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{1 + (\cos^2 x + \sin^2 x - 2 \sin x \cos x)} dx \quad 1M$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{1 + (\cos x - \sin x)^2} dx \quad 1A$$

$$\text{Let } u = \cos x - \sin x. \text{ Then } du = (-\sin x - \cos x) dx. \quad 1M$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} dx &= \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{1 + (\cos x - \sin x)^2} dx \\ &= - \int_1^0 \frac{1}{1 + u^2} du \quad 1A \end{aligned}$$

$$\begin{aligned} &= \int_0^1 \frac{du}{1 + u^2} \\ &= \frac{\pi}{4} \quad 1A \end{aligned}$$

$$2. (a) \text{ Let } u = a - x. \text{ Then } du = -dx. \quad 1M$$

$$\begin{aligned} \int_0^a f(a - x) dx &= - \int_a^0 f(u) du \\ &= \int_0^a f(u) du \quad 1M \end{aligned}$$

$$= \int_0^a f(x) dx \quad 1$$

$$\begin{aligned} (b) (i) \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta &= \int_0^{\frac{\pi}{2}} \frac{\cos(\frac{\pi}{2} - \theta)}{\sin(\frac{\pi}{2} - \theta) + \cos(\frac{\pi}{2} - \theta)} d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta \quad 1 \end{aligned}$$

$$\begin{aligned} (ii) \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta &= \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta + \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta \right] \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \\ &= \frac{1}{2} \left[\theta \right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4} \quad 1A \end{aligned}$$

$$(c) \text{ Let } x = \sin \theta. \text{ Then } dx = \cos \theta d\theta. \quad 1M$$

$$\begin{aligned} \int_0^1 \frac{dx}{x + \sqrt{1 - x^2}} &= \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \sqrt{1 - \sin^2 \theta}} d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta \\ &= \frac{\pi}{4} \quad 1A \end{aligned}$$

3. (a) Let $x = \sqrt{3} \tan \theta$. Then $dx = \sqrt{3} \sec^2 \theta d\theta$. 1M

$$\begin{aligned} \int_0^1 \frac{1}{x^2 + 3} dx &= \int_0^{\frac{\pi}{6}} \frac{\sqrt{3} \sec^2 \theta}{3 \tan^2 \theta + 3} d\theta \\ &= \frac{\sqrt{3}}{3} \int_0^{\frac{\pi}{6}} d\theta \\ &= \frac{\sqrt{3}}{3} \left[\theta \right]_0^{\frac{\pi}{6}} \\ &= \frac{\sqrt{3}\pi}{18} \end{aligned}$$

1M
1A

(b) (i) $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}}$

$$\begin{aligned} &= \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} \\ &= \frac{\cos 2\theta}{1} \\ &= \cos 2\theta \end{aligned}$$

1

- (ii) Let $t = \tan \theta$. Then $dt = \sec^2 \theta d\theta$ and $d\theta = \frac{1}{1+t^2} dt$. 1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 2} d\theta &= \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} + 2} \cdot \frac{1}{1+t^2} dt \\ &= \int_0^1 \frac{1}{t^2 + 3} dt \\ &= \frac{\sqrt{3}\pi}{18} \end{aligned}$$

1M
1A

- (c) Let $u = \frac{\pi}{4} - \theta$. Then $du = -d\theta$. 1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 2} d\theta &= - \int_{\frac{\pi}{4}}^0 \frac{1}{\cos \left[\frac{\pi}{2} - 2u \right] + 2} du \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2u + 2} du \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + 2} d\theta \end{aligned}$$

1

$$\begin{aligned}
\text{(d)} \quad & \int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + \cos 2\theta + 4}{(\sin 2\theta + 2)(\cos 2\theta + 2)} d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 2}{(\sin 2\theta + 2)(\cos 2\theta + 2)} d\theta + \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 2}{(\sin 2\theta + 2)(\cos 2\theta + 2)} d\theta & 1\text{M} \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 2} d\theta + \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + 2} d\theta \\
&= 2 \int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 2} d\theta \\
&= 2 \times \frac{\sqrt{3}\pi}{18} & 1\text{M} \\
&= \frac{\sqrt{3}\pi}{9} & 1\text{A}
\end{aligned}$$

4. (a) Let $u = \pi - x$. Then $du = -dx$. 1M

$$\begin{aligned}
\int_0^{\pi} x f(x) dx &= - \int_{\pi}^0 (\pi - u) f(\pi - u) du \\
&= \int_0^{\pi} (\pi - u) f(u) du & 1\text{M} \\
&= \int_0^{\pi} (\pi - x) f(x) dx & 1 \\
\int_0^{\pi} x f(x) dx &= \frac{1}{2} \left[\int_0^{\pi} x f(x) dx + \int_0^{\pi} (\pi - x) f(x) dx \right] \\
&= \frac{\pi}{2} \int_0^{\pi} f(x) dx & 1
\end{aligned}$$

(b) Let $u = \cos x$. Then $du = -\sin x dx$. 1M

$$\begin{aligned}
\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx &= - \int_1^{-1} \frac{du}{1 + u^2} \\
&= \int_{-1}^1 \frac{du}{1 + u^2}
\end{aligned}$$

Let $u = \tan \theta$. Then $du = \sec^2 \theta d\theta$. 1M

$$\begin{aligned}
\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1 + \tan^2 \theta} d\theta & 1\text{A} \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \\
&= \left[\theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{\pi}{2} & 1\text{A}
\end{aligned}$$

$$(c) \text{ Let } f(x) = \frac{\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right)}{3 + \cos 2x}.$$

$$f(\pi - x) = \frac{\cos\left(\frac{\pi}{6} - \pi + x\right) - \cos\left(\frac{\pi}{6} + \pi - x\right)}{3 + \cos(2\pi - 2x)} = \frac{-\cos\left(\frac{\pi}{6} + x\right) + \cos\left(\frac{\pi}{6} - x\right)}{3 + \cos 2x} = f(x) \quad 1M$$

$$\int_0^\pi x \left[\frac{\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right)}{3 + \cos 2x} \right] dx$$

$$= \frac{\pi}{2} \int_0^\pi \frac{\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right)}{3 + \cos 2x} dx \quad 1M$$

$$= \frac{\pi}{2} \int_0^\pi \frac{-2 \sin \frac{\pi}{6} \sin(-x)}{3 + \cos 2x} dx \quad 1M$$

$$= \frac{\pi}{2} \int_0^\pi \frac{\sin x}{3 + (2 \cos^2 x - 1)} dx$$

$$= \frac{\pi}{4} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx$$

$$= \frac{\pi}{4} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{8} \quad 1A$$

$$\begin{aligned}
 5. \quad (a) \quad (i) \quad \int \sin^4 x \, dx &= \int \sin^3 x \sin x \, dx \\
 &= -\sin^3 x \cos x + 3 \int \sin^2 x \cos^2 x \, dx & 1M
 \end{aligned}$$

$$= -\sin^3 x \cos x + 3 \int \sin^2 x (1 - \sin^2 x) \, dx$$

$$= -\sin^3 x \cos x + 3 \int \sin^2 x \, dx - 3 \int \sin^4 x \, dx$$

$$4 \int \sin^4 x \, dx = -\sin^3 x \cos x + 3 \int \sin^2 x \, dx$$

$$\int \sin^4 x \, dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \int \sin^2 x \, dx \quad 1$$

$$(ii) \quad \int_0^{2\pi} \sin^4 x \, dx = -\frac{1}{4} \left[\cos x \sin^3 x \right]_0^{2\pi} + \frac{3}{4} \int_0^{2\pi} \sin^2 x \, dx$$

$$= \frac{3}{8} \int_0^{2\pi} (1 - \cos 2x) \, dx \quad 1M$$

$$= \frac{3}{8} \left[x - \frac{\sin 2x}{2} \right]_0^{2\pi} \quad 1M$$

$$= \frac{3\pi}{4} \quad 1A$$

$$(b) \quad (i) \quad \text{Let } u = a - x. \text{ Then } du = -dx. \quad 1M$$

$$\int_0^a x f(x) \, dx = - \int_a^0 (a - u) f(u) \, du \quad 1M$$

$$= \int_0^a (a - u) f(u) \, du$$

$$= a \int_0^x f(x) \, dx - \int_0^a x f(x) \, dx$$

$$2 \int_0^a x f(x) \, dx = a \int_0^a f(x) \, dx$$

$$\int_0^a x f(x) \, dx = \frac{a}{2} \int_0^a f(x) \, dx \quad 1$$

$$(ii) \quad \text{Put } f(x) = \sin^4 x. \text{ Then } f(2\pi - x) = \sin^4(2\pi - x) = \sin^4 x = f(x). \quad 1M$$

$$\int_0^{2\pi} x \sin^4 x \, dx = \frac{2\pi}{2} \int_0^{2\pi} \sin^4 x \, dx \quad 1M$$

$$= \pi \left(\frac{3\pi}{4} \right)$$

$$= \frac{3\pi^2}{4} \quad 1A$$