M2REG-DEFINT-2223-ASM-SET 3-MATH
Suggested solutions

1. $\displaystyle\int_3^5 (x+1)e^{\frac{x-1}{2}}\,dx = 2\left[(x+1)e^{\frac{x-1}{2}}\right]_3^5 - 2\int_3^5 e^{\frac{x-1}{2}}\,dx$ 1M+1A

$\displaystyle = 2(6e^2 - 4e) - 4\left[e^{\frac{x-1}{2}}\right]_3^5$

$= 8e^2 - 4e$ 1A

2. $\displaystyle\int_{-1}^0 (3-x)e^{-3x}\,dx = -\frac{1}{3}\left[(3-x)e^{-3x}\right]_{-1}^0 - \frac{1}{3}\int_{-1}^0 e^{-3x}\,dx$ 1M+1A

$\displaystyle = -\frac{1}{3}(3-4e^3) + \frac{1}{9}\left[e^{-3x}\right]_{-1}^0$

$\displaystyle = -\frac{8}{9} + \frac{11}{9}e^3$ 1A

3. (a) $\displaystyle\int x\cos kx\,dx = \frac{x\sin kx}{k} - \frac{1}{k}\int \sin kx\,dx$ 1M

$\displaystyle = \frac{x\sin kx}{x} + \frac{\cos kx}{k^2} + \text{constant}$ 1A

(b) $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\cos 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} x(\cos 3x + \cos x)\,dx$ 1M

$\displaystyle = \frac{1}{2}\int_0^{\frac{\pi}{2}} x\cos 3x\,dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} x\cos x\,dx$

$\displaystyle = \frac{1}{2}\left[\frac{x\sin 3x}{3} + \frac{\cos 3x}{9}\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\left[x\sin x + \cos x\right]_0^{\frac{\pi}{2}}$ 1M

$\displaystyle = \frac{\pi}{6} - \frac{5}{9}$ 1A

4. $\displaystyle\int_2^8 \ln\frac{x}{2}\,dx = \left[x\ln\frac{x}{2}\right]_2^8 - \int_2^8 dx$ 1M+1A

$\displaystyle = 16\ln 2 - \left[x\right]_2^8$

$= 16\ln 2 - 6$ 1A

5. $\displaystyle\int_2^4 \left(x - \frac{1}{x^2}\right)\ln x\,dx = \left[\left(\frac{x^2}{2} + \frac{1}{x}\right)\ln x\right]_2^4 - \int_2^4 \left(\frac{x}{2} + \frac{1}{x^2}\right)dx$ 1M+1A

$\displaystyle = 14\ln 2 - \left[\frac{x^2}{4} - \frac{1}{x}\right]_2^4$

$\displaystyle = 14\ln 2 - \frac{13}{4}$ 1A

6. (a) Let $u = \ln x$. Then $\mathrm{d}u = \dfrac{\mathrm{d}x}{x}$. 1M

When $x = 1$, $u = 0$; when $x = e$, $u = 1$.

$$\int_1^e \frac{\ln x}{x}\,\mathrm{d}x = \int_0^1 u\,\mathrm{d}u \quad \text{1M}$$

$$= \left[\frac{u^2}{2}\right]_0^1 \quad \text{1A}$$

$$= \frac{1}{2} \quad \text{1A}$$

(b) $$\int_1^e \frac{(1+x^2)\ln x}{x}\,\mathrm{d}x = \int_1^e \frac{\ln x}{x}\,\mathrm{d}x + \int_1^e x\ln x\,\mathrm{d}x$$

$$= \frac{1}{2} + \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)\mathrm{d}x \quad \text{1M+1M}$$

$$= \frac{1}{2} + \frac{e^2}{2} - 0 - \frac{1}{2}\int_1^e x\,\mathrm{d}x$$

$$= \frac{1+e^2}{2} - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e$$

$$= \frac{1+e^2}{2} - \frac{e^2-1}{4}$$

$$= \frac{3+e^2}{4} \quad \text{1A}$$

7. (a) $$\int_1^3 e^{2\ln x}\ln x\,\mathrm{d}x = \int_1^3 x^2\ln x\,\mathrm{d}x$$

$$= \frac{1}{3}\left[x^3\ln x\right]_1^3 - \frac{1}{3}\int_1^3 x^2\,\mathrm{d}x \quad \text{1M}$$

$$= 9\ln 3 - \frac{1}{9}\left[x^3\right]_1^3 \quad \text{1A}$$

$$= 9\ln 3 - \frac{26}{9} \quad \text{1A}$$

(b) Let $u = \dfrac{1}{x}$. Then $\mathrm{d}u = -\dfrac{1}{x^2}\,\mathrm{d}x$. 1M

$$\int_{\frac{1}{3}}^1 \frac{e^{-2\ln x}\ln x}{x^2}\,\mathrm{d}x = -\int_3^1 e^{-2\ln\frac{1}{u}}\ln\frac{1}{u}\,\mathrm{d}u$$

$$= -\int_1^3 e^{2\ln u}\ln u\,\mathrm{d}u$$

$$= -\left(9\ln 3 - \frac{26}{9}\right)$$

$$= \frac{26}{9} - 9\ln 3 \quad \text{1A}$$

8. $$\int_1^{e^{\frac{\pi}{2}}} \sin(\ln x)\,dx = \Big[x\sin(\ln x)\Big]_1^{e^{\frac{\pi}{2}}} - \int_1^{e^{\frac{\pi}{2}}} \cos(\ln x)\,dx \quad \text{1M}$$

$$= e^{\frac{\pi}{2}} - \Big[x\cos(\ln x)\Big]_1^{e^{\frac{\pi}{2}}} - \int_1^{e^{\frac{\pi}{2}}} \sin(\ln x)\,dx \quad \text{1M}$$

$$= e^{\frac{\pi}{2}} + 1 - \int_1^{e^{\frac{\pi}{2}}} \sin(\ln x)\,dx$$

$$2\int_1^{e^{\frac{\pi}{2}}} \sin(\ln x)\,dx = e^{\frac{\pi}{2}} + 1$$

$$\int_1^{e^{\frac{\pi}{2}}} \sin(\ln x)\,dx = \frac{e^{\frac{\pi}{2}}}{2} + \frac{1}{2} \quad \text{1A}$$

9. (a) Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta\,d\theta$. 1M

$$\int_0^{\sqrt{3}} \frac{1}{x^2+9}\,dx = \int_0^{\frac{\pi}{6}} \frac{3\sec^2\theta}{9\tan^2\theta+9}\,d\theta \quad \text{1A}$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{6}} d\theta$$

$$= \frac{1}{3}\Big[\theta\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{18} \quad \text{1A}$$

(b) $$\int_0^{\sqrt{3}} \ln\left(x^2+9\right)dx = \Big[x\ln\left(x^2+9\right)\Big]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{2x^2}{x^2+9}\,dx \quad \text{1M}$$

$$= \sqrt{3}\ln 12 - 2\int_0^{\sqrt{3}} \left(1 - \frac{9}{x^2+9}\right)dx \quad \text{1M}$$

$$= \sqrt{3}\ln 12 - 2\int_0^{\sqrt{3}} dx + 18\int_0^{\sqrt{3}} \frac{1}{x^2+9}\,dx$$

$$= \sqrt{3}\ln 12 - 2\Big[x\Big]_0^{\sqrt{3}} + 18\left(\frac{\pi}{18}\right) \quad \text{1M}$$

$$= \sqrt{3}\ln 12 - 2\sqrt{3} + \pi \quad \text{1A}$$

10. Let $u = x^2$. Then $du = 2x\,dx$. 1M

$$\int_{\sqrt{\frac{\pi}{3}}}^{\sqrt{\pi}} 2x^3\sin x^2\,dx = \int_{\frac{\pi}{3}}^{\pi} u\sin u\,du$$

$$= \Big[-u\cos u\Big]_{\frac{\pi}{3}}^{\pi} + \int_{\frac{\pi}{3}}^{\pi} \cos u\,du \quad \text{1M}$$

$$= \frac{7\pi}{6} + \Big[\sin u\Big]_{\frac{\pi}{3}}^{\pi}$$

$$= \frac{7\pi}{6} - \frac{\sqrt{3}}{2} \quad \text{1A}$$

11. Let $u = x^3 - 1$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$. 1M

$$\int_{-1}^{1}(3x^5 - 3x^2)e^{x^3-1}\,\mathrm{d}x = \int_{-2}^{0} ue^u\,\mathrm{d}u$$

$$= \Big[ue^u\Big]_{-2}^{0} - \int_{-2}^{0} e^u\,\mathrm{d}u \quad \text{1M}$$

$$= 2e^{-2} - \Big[e^u\Big]_{-2}^{0}$$

$$= 3e^{-1} - 1 \quad \text{1A}$$

12. $$\int_0^{\frac{\pi}{6}} x^2\cos 3x\,\mathrm{d}x = \left[\frac{x^2\sin 3x}{3}\right]_0^{\frac{\pi}{6}} - \frac{2}{3}\int_0^{\frac{\pi}{6}} x\sin 3x\,\mathrm{d}x \quad \text{1M+1A}$$

$$= \frac{\pi^2}{108} + \frac{2}{9}\Big[x\cos 3x\Big]_0^{\frac{\pi}{6}} - \frac{2}{9}\int_0^{\frac{\pi}{6}} \cos 3x\,\mathrm{d}x$$

$$= \frac{\pi^2}{108} + 0 - \frac{2}{27}\Big[\sin 3x\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi^2}{108} - \frac{2}{27} \quad \text{1A}$$

13. $$\int_0^3 \left(\frac{x}{e^{2x}}\right)^2 \mathrm{d}x = \int_0^3 x^2e^{-4x}\,\mathrm{d}x$$

$$= -\frac{1}{4}\Big[x^2e^{-4x}\Big]_0^3 + \frac{1}{2}\int_0^3 xe^{-4x}\,\mathrm{d}x \quad \text{1M+1A}$$

$$= -\frac{9e^{-12}}{4} - \frac{1}{8}\Big[xe^{-4x}\Big]_0^3 + \frac{1}{8}\int_0^3 e^{-4x}\,\mathrm{d}x$$

$$= -\frac{9e^{-12}}{4} - \frac{3e^{-12}}{8} - \frac{1}{32}\Big[e^{-4x}\Big]_0^3$$

$$= -\frac{85}{32e^{12}} + \frac{1}{32} \quad \text{1A}$$

14. $$\int_0^{\frac{\pi}{4}} e^x\sin 2x\,\mathrm{d}x = \Big[e^x\sin 2x\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x(2\cos 2x)\,\mathrm{d}x \quad \text{1M+1M}$$

$$= (e^{\frac{\pi}{4}} - 0) - 2\left[\Big[e^x\cos 2x\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x(-2\sin 2x)\,\mathrm{d}x\right] \quad \text{1M}$$

$$5\int_0^{\frac{\pi}{4}} e^x\sin 2x\,\mathrm{d}x = e^{\frac{\pi}{4}} - 2(0-1) \quad \text{1M}$$

$$\int_0^{\frac{\pi}{4}} e^x\sin 2x\,\mathrm{d}x = \frac{1}{5}(e^{\frac{\pi}{4}} + 2) \quad \text{1A}$$

END OF PAPER