

M2REG-INDINT-2223-ASM-SET 6-MATH

Suggested solutions

$$\begin{aligned}
 1. \int \frac{x^{\frac{3}{2}} - 8}{\sqrt{x} - 2} dx &= \int \frac{(x^{\frac{1}{2}} - 2)(x + 2x^{\frac{1}{2}} + 4)}{x^{\frac{1}{2}} - 2} dx \\
 &= \int (x + 2x^{\frac{1}{2}} + 4) dx && 1M \\
 &= \frac{1}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}} + 4x + \text{constant} && 1A
 \end{aligned}$$

$$\begin{aligned}
 2. \int \frac{3e^{2x} + 5e^x - 2}{e^x + 2} dx &= \int \frac{(3e^x - 1)(e^x + 2)}{e^x + 2} dx \\
 &= \int (3e^x - 1) dx && 1M \\
 &= 3e^x - x + \text{constant} && 1A
 \end{aligned}$$

$$\begin{aligned}
 3. \text{ Let } u = \log_2 x. \text{ Then } du &= \frac{1}{x \ln 2} dx. && 1M \\
 \int \frac{3(\log_2 x)^2 + 2 \log_2 x - 4}{x} dx &= \int (\ln 2)(3u^2 + 2u - 4) du && 1A \\
 &= (\ln 2)(u^3 + u^2 - 4u) + \text{constant} \\
 &= (\ln 2)[(\log_2 x)^3 + (\log_2 x)^2 - 4 \log_2 x] + \text{constant} && 1A
 \end{aligned}$$

$$\begin{aligned}
 4. \text{ Let } u = e^x + 1. \text{ Then } du &= e^x dx. && 1M \\
 \int \frac{e^x}{e^{2x} + 2e^x + 1} dx &= \int \frac{e^x}{(e^x + 1)^2} dx \\
 &= \int \frac{du}{u^2} && 1A \\
 &= -\frac{1}{u} + \text{constant} \\
 &= -\frac{1}{e^x + 1} + \text{constant} && 1A
 \end{aligned}$$

$$\begin{aligned}
 5. \text{ Let } u = \ln x. \text{ Then } du &= \frac{1}{x} dx. && 1M \\
 \int \frac{\sqrt{\ln \sqrt{x}}}{x} dx &= \int \frac{\sqrt{\frac{1}{2} \ln x}}{x} dx \\
 &= \frac{1}{\sqrt{2}} \int u^{\frac{1}{2}} du && 1M \\
 &= \frac{\sqrt{2}}{3} u^{\frac{3}{2}} + \text{constant} \\
 &= \frac{\sqrt{2}}{3} (\ln x)^{\frac{3}{2}} + \text{constant}
 \end{aligned}$$

6. Let $u = 1 - e^{-5x}$. Then $du = 5e^{-5x} dx$. 1M

$$\begin{aligned}\int \frac{dx}{e^{5x} - 1} &= \int \frac{e^{-5x}}{1 - e^{-5x}} \\ &= \frac{1}{5} \int \frac{du}{u} & 1M \\ &= \frac{1}{5} \ln |u| + \text{constant} \\ &= \frac{1}{5} \ln |1 - e^{-5x}| + \text{constant} & 1A\end{aligned}$$

Alternative solution

Let $u = e^{5x} - 1$. Then $du = 5e^{5x} dx$. 1M

$$\begin{aligned}\int \frac{dx}{e^{5x} - 1} &= \int \frac{1 - e^{5x} + e^{5x}}{e^{5x} - 1} dx \\ &= \int \frac{1 - e^{5x}}{e^{5x} - 1} dx + \int \frac{e^{5x}}{e^{5x} - 1} dx \\ &= - \int dx + \frac{1}{5} \int \frac{du}{u} & 1M \\ &= -x + \frac{1}{5} \ln |u| + \text{constant} \\ &= -x + \frac{1}{5} \ln |e^{5x} - 1| + \text{constant} & 1A\end{aligned}$$

7. Let $u = \sqrt{x} - 1$. Then $du = \frac{1}{2\sqrt{x}} dx$. 1M

$$\begin{aligned}\int \sqrt{\sqrt{x} - 1} dx &= \int 2\sqrt{x} \sqrt{\sqrt{x} - 1} \cdot \frac{1}{2\sqrt{x}} dx \\ &= \int 2(u + 1)\sqrt{u} du \\ &= \int (2u^{\frac{3}{2}} + 2u^{\frac{1}{2}}) du & 1A \\ &= \frac{4}{5} u^{\frac{5}{2}} + \frac{4}{3} u^{\frac{3}{2}} + \text{constant} \\ &= \frac{4}{5} (\sqrt{x} - 1)^{\frac{5}{2}} + \frac{4}{3} (\sqrt{x} - 1)^{\frac{3}{2}} + \text{constant} & 1A\end{aligned}$$

8. (a) $\int \frac{x+1}{x+3} dx = \int \left(1 - \frac{2}{x+3}\right) dx$ 1M
 $= x - 2 \ln |x+3| + \text{constant}$ 1A

(b) Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} dx$.

$$\begin{aligned}\int \frac{\sqrt{x} + 1}{x + 3\sqrt{x}} dx &= 2 \int \frac{\sqrt{x} + 1}{\sqrt{x} + 3} \cdot \frac{1}{2\sqrt{x}} dx \\ &= 2 \int \frac{u + 1}{u + 3} du & 1M \\ &= 2(u - 2 \ln |u + 3|) + \text{constant} \\ &= 2\sqrt{x} - 4 \ln(\sqrt{x} + 3) + \text{constant} & 1A\end{aligned}$$

9. Let $u = 3 + \sin(\ln x)$. Then $du = \frac{\cos(\ln x)}{x} dx$. 1M

$$\int \frac{\cos(\ln x)}{x[3 + \sin(\ln x)]} dx = \int \frac{du}{u} \quad 1A$$

$$= \ln |3 + \sin(\ln x)| + \text{constant}$$

$$= \ln(3 + \sin(\ln x)) + \text{constant} \quad 1A$$

10. Let $u = \sin x$. Then $du = \cos x dx$. 1M

$$\int \frac{\cos^5 x}{\sin^6 x} dx = \int \frac{(1 - u^2)^2}{u^6} du$$

$$= \int (u^{-6} - 2u^{-4} + u^{-2}) du \quad 1A$$

$$= -\frac{1}{5u^5} + \frac{2}{3u^3} - \frac{1}{u} + \text{constant}$$

$$= -\frac{1}{5 \sin^5 x} + \frac{2}{3 \sin^3 x} - \frac{1}{\sin x} + \text{constant} \quad 1A$$

11. Let $u = \tan x$. Then $du = \sec^2 x dx$. 1M

$$\int \frac{\sec^8 x}{\tan^3 x} dx = \int \frac{(1 + \tan^2 x)^3}{\tan^3 x} \cdot \sec^2 x dx$$

$$= \int \frac{(1 + u^2)^3}{u^3} du$$

$$= \int (u^{-3} + 3u^{-1} + 3u + u^3) du \quad 1A$$

$$= -\frac{1}{2 \tan^2 x} + 3 \ln |\tan x| + \frac{3}{2} \tan^2 x + \frac{1}{4} \tan^4 x + \text{constant} \quad 1A$$

12. $\int \frac{2 \sin^4 2x}{1 - \cos 2x} dx = \int \frac{2 \sin^4 2x (1 + \cos 2x)}{1 - \cos^2 2x} dx$ 1M

$$= \int 2 \sin^2 2x (1 + \cos 2x) dx$$

$$= \int (1 - \cos 4x)(1 + \cos 2x) dx \quad 1M$$

$$= \int (1 - \cos 4x + \cos 2x - \cos 4x \cos 2x) dx$$

$$= \int \left[1 - \cos 4x + \cos 2x - \frac{1}{2}(\cos 6x + \cos 2x) \right] dx \quad 1M$$

$$= x + \frac{\sin 2x}{4} - \frac{\sin 4x}{4} - \frac{\sin 6x}{12} + \text{constant} \quad 1A$$

13. Let $u = 5 - 11x^2$. Then $du = -22x dx$. 1M

$$\int x \sqrt{5 - 11x^2} dx = -\frac{1}{22} \int \sqrt{u} du \quad 1A$$

$$= -\frac{1}{33} u^{\frac{3}{2}} + \text{constant}$$

$$= -\frac{1}{33} (5 - 11x^2)^{\frac{3}{2}} + \text{constant} \quad 1A$$

$$14. \quad (a) \quad x^2 - 4x \cos \frac{\pi}{12} + 4 = x^2 - 4x \cos \frac{\pi}{12} + 4 \cos^2 \frac{\pi}{12} + 4 \sin^2 \frac{\pi}{12} \\ = \left(x - 2 \cos \frac{\pi}{12}\right)^2 + 4 \sin^2 \frac{\pi}{12} \quad 1A$$

$$(b) \quad \text{Let } x - 2 \cos \frac{\pi}{12} = 2 \sin \frac{\pi}{12} \tan \theta. \text{ Then } dx = 2 \sin \frac{\pi}{12} \sec^2 \theta d\theta. \quad 1M$$

$$\int \frac{\sin \frac{\pi}{12}}{x^2 - 4 \cos \frac{\pi}{12} x + 4} dx = \int \frac{\sin \frac{\pi}{12}}{\left(x - 2 \cos \frac{\pi}{12}\right)^2 + 4 \sin^2 \frac{\pi}{12}} dx \\ = \int \frac{\sin \frac{\pi}{12}}{4 \sin^2 \frac{\pi}{12} \tan^2 \theta + 4 \sin^2 \frac{\pi}{12}} \cdot 2 \sin \frac{\pi}{12} \sec^2 \theta d\theta \\ = \frac{1}{2} \int d\theta \quad 1A$$

$$= \frac{1}{2} \theta + \text{constant} \\ = \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \csc \frac{\pi}{12} - \cot \frac{\pi}{12} \right) + \text{constant} \\ = \frac{1}{2} \tan^{-1} \left(\frac{\sqrt{6} + \sqrt{2}}{2} x - 2 - \sqrt{3} \right) + \text{constant} \quad 1A$$

$$15. \quad \int \sec^2 x \ln(\sin x) dx = \tan x \ln(\sin x) - \int \tan x \cdot \frac{\cos x}{\sin x} dx \quad 1M \\ = \tan x \ln(\sin x) - \int dx \\ = \tan x \ln(\sin x) - x + \text{constant} \quad 1A$$

$$16. \quad \int e^{\sqrt{x}} dx = \int \sqrt{x} \cdot \frac{e^{\sqrt{x}}}{\sqrt{x}} dx \\ \text{Let } u = \sqrt{x}. \text{ Then } du = \frac{1}{2\sqrt{x}} dx. \\ \int e^{\sqrt{x}} dx = 2 \int u \cdot e^u du \\ = 2ue^u - 2 \int e^u du \quad 1M \\ = 2ue^u - 2e^u + \text{constant} \\ = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + \text{constant} \quad 1A$$

17. (a) (i) Let $u = \ln x$. Then $du = \frac{1}{x} dx$. 1M

$$\begin{aligned}\int \frac{(\ln x)^n}{x} dx &= \int u^n dx \\ &= \frac{u^{n+1}}{n+1} + \text{constant} \\ &= \frac{(\ln x)^{n+1}}{n+1} + \text{constant}\end{aligned}$$

1A

(ii) $\int x^a (\ln x)^n dx = \int \left(x^{a+1} \times \frac{(\ln x)^n}{x} \right) dx$ 1M

$$\begin{aligned}&= \frac{x^{a+1} (\ln x)^{n+1}}{n+1} - \int \left[(a+1)x^a \times \frac{(\ln x)^{n+1}}{n+1} \right] dx \\ &= \frac{x^{a+1} (\ln x)^{n+1}}{n+1} - \frac{a+1}{n+1} \int x^a (\ln x)^{n+1} dx\end{aligned}$$

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(b) (i) $\int x^a dx = \frac{x^{a+1}}{a+1} + \text{constant}$ 1A

(ii) Using the result of (a)(ii), we have

$$\begin{aligned}\int x^a (\ln x)^{n+1} dx &= \frac{x^{a+1} (\ln x)^{n+1}}{a+1} - \frac{n+1}{a+1} \int x^a (\ln x)^n dx . \\ \int x^a \ln x dx &= \frac{x^{a+1} (\ln x)}{a+1} - \frac{1}{a+1} \int x^a dx \\ &= \frac{x^{a+1} \ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} + \text{constant}\end{aligned}$$

1M

1A

$$\begin{aligned}\int x^a (\ln x)^2 dx &= \frac{x^{a+1} (\ln x)^2}{a+1} - \frac{2}{a+1} \int x^a (\ln x) dx \\ &= \frac{x^{a+1} (\ln x)^2}{a+1} - \frac{2}{a+1} \left[\frac{x^{a+1} \ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} \right] + \text{constant} \\ &= \frac{x^{a+1} (\ln x)^2}{a+1} - \frac{2x^{a+1} \ln x}{(a+1)^2} + \frac{2x^{a+1}}{(a+1)^3} + \text{constant}\end{aligned}$$

1M

1A