

M2REG-INDINT-2223-ASM-SET 4-MATH

Suggested solutions

1. Let $u = x^2 + 1$. Then $du = 2x \, dx$. 1M

$$\begin{aligned} \int \frac{x(x^2 - 1)}{\sqrt{x^2 + 1}} \, dx &= \frac{1}{2} \int \frac{u - 2}{\sqrt{u}} \, du \\ &= \frac{1}{2} \int (u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}) \, du && 1A \\ &= \frac{1}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + \text{constant} \\ &= \frac{1}{3} (x^2 + 1)^{\frac{3}{2}} - 2(x^2 + 1)^{\frac{1}{2}} + \text{constant} && 1A \end{aligned}$$

2. (a) $\frac{A}{3x+2} + \frac{B}{5x-1} = \frac{A(5x-1) + B(3x+2)}{(3x+2)(5x-1)}$
 $= \frac{(5A+3B)x + (-A+2B)}{(3x+2)(5x-1)}$

We have $5A + 3B = 12$ and $-A + 2B = -5$. 1M

Solving, we have $A = 3$ and $B = -1$. 1A+1A

(b) $\int \frac{12x-5}{(3x+2)(5x-1)} \, dx = \int \left(\frac{3}{3x+2} - \frac{1}{5x-1} \right) \, dx$ 1M
 $= \ln |3x+2| - \frac{1}{5} \ln |5x-1| + \text{constant}$ 1A

3. (a) $P + \frac{Q}{x+4} + \frac{R}{2x-5} = \frac{P(x+4)(2x-5) + Q(2x-5) + R(x+4)}{(x+4)(2x-5)}$
 $= \frac{2Px^2 + (3P+2Q+R)x + (-20P-5Q+4R)}{(x+4)(2x-5)}$

We have $2P = 4$, $3P + 2Q + R = 11$ and $-20P - 5Q + 4R = -59$. 1M

Solving, we have $P = 2$, $Q = 3$ and $R = -1$. 1A+1A

(b) $\int \frac{4x^2 + 11x - 59}{2x^2 + 3x - 20} \, dx = \int \frac{4x^2 + 11x - 59}{(x+4)(2x-5)} \, dx$
 $= \int \left(2 + \frac{3}{x+4} - \frac{1}{2x-5} \right) \, dx$ 1M
 $= 2x + 3 \ln |x+4| - \frac{1}{2} \ln |2x-5| + \text{constant}$ 1A

$$4. \quad (a) \quad \frac{A}{1-x} + \frac{B}{1+x} = \frac{A(1+x) + B(1-x)}{(1-x)(1+x)} \\ = \frac{(A-B)x + (A+B)}{1-x^2}$$

We have $A - B = 0$ and $A + B = 1$.

Solving, we have $A = \frac{1}{2}$ and $B = \frac{1}{2}$.

(b) Let $u = \cos x$. Then $du = -\sin x \, dx$.

$$\int \frac{1}{\sin x} \, dx = \int \frac{\sin x}{1 - \cos^2 x} \, dx \\ = - \int \frac{du}{1 - u^2} \\ = -\frac{1}{2} \int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du \\ = \frac{1}{2} \ln |1-u| - \frac{1}{2} \ln |1+u| + \text{constant} \\ = \frac{1}{2} \ln \left| \frac{1 - \cos x}{1 + \cos x} \right| + \text{constant}$$

(c) Let $3x = \sin \theta$. Then $3 \, dx = \cos \theta \, d\theta$.

$$\int \frac{dx}{x\sqrt{1-9x^2}} = \frac{1}{3} \int \frac{\cos \theta}{\frac{\sin \theta}{3} \sqrt{1 - \sin^2 \theta}} \, d\theta \\ = \int \frac{d\theta}{\sin \theta} \\ = \frac{1}{2} \ln \left| \frac{1 - \cos \theta}{1 + \cos \theta} \right| + \text{constant} \\ = \frac{1}{2} \ln \left| \frac{1 - \sqrt{1-9x^2}}{1 + \sqrt{1-9x^2}} \right| + \text{constant}$$

$$5. \quad \int (x+2) \sin x \, dx = -(x+2) \cos x + \int \cos x \, dx$$

$$= -(x+2) \cos x + \sin x + \text{constant}$$

$$6. \quad \int x \cos(2x+1) \, dx = \frac{1}{2} x \sin(2x+1) - \frac{1}{2} \int \sin(2x+1) \, dx$$

$$= \frac{1}{2} x \sin(2x+1) + \frac{1}{4} \cos(2x+1) + \text{constant}$$

$$7. \quad \int x^2 \cos(3x-4) \, dx = \frac{x^2 \sin(3x-4)}{3} - \frac{2}{3} \int x \sin(3x-4) \, dx$$

$$= \frac{x^2 \sin(3x-4)}{3} + \frac{2x \cos(3x-4)}{9} - \frac{2}{9} \int \cos(3x-4) \, dx$$

$$= \frac{x^2 \sin(3x-4)}{3} + \frac{2x \cos(3x-4)}{9} - \frac{2 \sin(3x-4)}{27} + \text{constant}$$

$$8. \quad \int x^{\frac{1}{4}} \ln x \, dx = \frac{4}{5} x^{\frac{5}{4}} \ln x - \frac{4}{5} x^{\frac{1}{4}} \, dx$$

$$= \frac{4}{5} x^{\frac{5}{4}} \ln x - \frac{16}{25} x^{\frac{5}{4}} + \text{constant}$$

$$9. \int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right) \ln x \, dx = \left(\frac{2}{3} x^{\frac{3}{2}} - 2x^{\frac{1}{2}} \right) \ln x - \int \left(\frac{2}{3} x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \right) dx$$

$$= \left(\frac{2}{3} x^{\frac{3}{2}} - 2x^{\frac{1}{2}} \right) \ln x - \frac{4}{9} x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + \text{constant}$$

1M

1A

$$10. \int x^5 (\ln x)^2 \, dx = \frac{x^6 (\ln x)^2}{6} - \frac{1}{3} \int x^5 \ln x \, dx$$

$$= \frac{x^6 (\ln x)^2}{6} - \frac{x^6 \ln x}{18} + \frac{1}{18} \int x^5 \, dx$$

$$= \frac{x^6 (\ln x)^2}{6} - \frac{x^6 \ln x}{18} + \frac{x^6}{108} + \text{constant}$$

1M

1M

1A

$$11. \int (x + e) e^x \, dx = (x + e) e^x - \int e^x \, dx$$

$$= (x + e) e^x - e^x + \text{constant}$$

$$= e^x (x + e - 1) + \text{constant}$$

1M

1A

$$12. \int x e^{\pi-x} \, dx = -x e^{\pi-x} + \int e^{\pi-x} \, dx$$

$$= -x e^{\pi-x} - e^{\pi-x} + \text{constant}$$

$$= -(x + 1) e^{\pi-x} + \text{constant}$$

1M

1A

$$13. \int (x^2 + e^x) e^x \, dx = \int x^2 e^x \, dx + \int e^{2x} \, dx$$

$$= x^2 e^x - 2 \int x e^x \, dx + \int e^{2x} \, dx$$

$$= x^2 e^x - 2x e^x + 2 \int e^x \, dx + \int e^{2x} \, dx$$

$$= x^2 e^x - 2x e^x + 2e^x + \frac{1}{2} e^{2x} + \text{constant}$$

1M

1M

1A

$$14. \int e^x \sin(x + 1) \, dx = e^x \sin(x + 1) - \int e^x \cos(x + 1) \, dx$$

$$= e^x \sin(x + 1) - e^x \cos(x + 1) - \int e^x \sin(x + 1) \, dx$$

1M

1M

$$2 \int e^x \sin(x + 1) \, dx = e^x [\sin(x + 1) - \cos(x + 1)] + \text{constant}$$

1M

$$\int e^x \sin(x + 1) \, dx = \frac{1}{2} e^x [\sin(x + 1) - \cos(x + 1)] + \text{constant}$$

1A

$$15. \int e^{-3x} \cos 4x \, dx = -\frac{e^{-3x} \cos 4x}{3} - \frac{4}{3} \int e^{-3x} \sin 4x \, dx$$

$$= -\frac{e^{-3x} \cos 4x}{3} + \frac{4e^{-3x} \sin 4x}{9} - \frac{16}{9} \int e^{-3x} \cos 4x \, dx$$

$$\frac{25}{9} \int e^{-3x} \cos 4x \, dx = -\frac{e^{-3x} \cos 4x}{3} + \frac{4e^{-3x} \sin 4x}{9} + \text{constant}$$

$$\int e^{-3x} \cos 4x \, dx = -\frac{3e^{-3x} \cos 4x}{25} + \frac{4e^{-3x} \sin 4x}{25} + \text{constant}$$

1M

1M

1M

1A

$$16. \quad \int x^2 \cos(\ln x) \, dx = \frac{x^3 \cos(\ln x)}{3} + \frac{1}{3} \int x^2 \sin(\ln x) \, dx \quad 1M$$

$$= \frac{x^3 \cos(\ln x)}{3} + \frac{x^3 \sin(\ln x)}{9} - \frac{1}{9} \int x^2 \cos(\ln x) \, dx \quad 1M$$

$$\frac{10}{9} \int x^2 \cos(\ln x) \, dx = \frac{x^3 \cos(\ln x)}{3} + \frac{x^3 \sin(\ln x)}{9} + \text{constant} \quad 1M$$

$$\int x^2 \cos(\ln x) \, dx = \frac{3x^3 \cos(\ln x)}{10} + \frac{x^3 \sin(\ln x)}{10} + \text{constant} \quad 1A$$

$$17. \quad (a) \quad I_n = x^n \left(-\frac{\cos ax}{a} \right) - \int (nx^{n-1}) \left(-\frac{\cos ax}{a} \right) dx \quad 1M+1M$$

$$= -\frac{x^n \cos ax}{a} + \frac{n}{a} \left[x^{n-1} \left(\frac{\sin ax}{a} \right) - \int [(n-1)x^{n-2}] \left(\frac{\sin ax}{a} \right) dx \right] \quad 1M$$

$$= -\frac{1}{a} x^n \cos ax + \frac{n}{a^2} x^{n-1} \sin ax - \frac{n(n-1)}{a^2} \int x^{n-2} \sin ax \, dx$$

$$= -\frac{1}{a} x^n \cos ax + \frac{n}{a^2} x^{n-1} \sin ax - \frac{n(n-1)}{a^2} I_{n-2} \quad 1$$

$$(b) \quad \int x \sin x \, dx = x(-\cos x) - \int (-\cos x) \, dx \quad 1M$$

$$= -x \cos x + \sin x + \text{constant} \quad 1A$$

$$(c) \quad \int x^5 \sin x \, dx$$

$$= -x^5 \cos x + 5x^4 \sin x - 5(4) \int x^3 \sin x \, dx \quad 1M$$

$$= -x^5 \cos x + 5x^4 \sin x - 20 \left[-x^3 \cos x + 3x^2 \sin x - 3(2) \int x \sin x \, dx \right] \quad 1M$$

$$= -x^5 + 5x^4 \sin x + 20x^3 \cos x - 60x^2 \sin x - 120x \cos x + 120 \sin x + \text{constant} \quad 1A$$