

M2REG-INDINT-2223-ASM-SET 3-MATH**Suggested solutions**

1. Let $x = 2\sqrt{2} \sin \theta$. Then $dx = 2\sqrt{2} \cos \theta d\theta$. 1M

$$\int \frac{dx}{\sqrt{8-x^2}} = \int \frac{2\sqrt{2} \cos \theta}{\sqrt{8-8\sin^2 \theta}} d\theta$$

$$= \int d\theta$$
1A

$$= \theta + \text{constant}$$

$$= \sin^{-1} \frac{x}{2\sqrt{2}} + \text{constant}$$
1A

2. Let $x = \sqrt{2} \tan \theta$. Then $dx = \sqrt{2} \sec^2 \theta d\theta$. 1M

$$\int \frac{dx}{2+x^2} = \int \frac{\sqrt{2} \sec^2 \theta}{2 \tan^2 \theta + 2} d\theta$$

$$= \frac{1}{\sqrt{2}} \int d\theta$$
1A

$$= \frac{1}{\sqrt{2}} \theta + \text{constant}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + \text{constant}$$
1A

3. Let $x = \sqrt{3} \sin \theta$. Then $dx = \sqrt{3} \cos \theta d\theta$. 1M

$$\int \sqrt{1-\frac{x^2}{3}} dx = \frac{1}{\sqrt{3}} \int \sqrt{3-3\sin^2 \theta} \cdot \sqrt{3} \cos \theta d\theta$$

$$= \sqrt{3} \int \cos^2 \theta d\theta$$
1A

$$= \frac{\sqrt{3}}{2} \int (1 + \cos 2\theta) d\theta$$
1M

$$= \frac{\sqrt{3}\theta}{2} + \frac{\sqrt{3}}{4} \sin 2\theta + \text{constant}$$

$$= \frac{\sqrt{3}\theta}{2} + \frac{\sqrt{3}}{2} \sin \theta \cos \theta + \text{constant}$$

$$= \frac{\sqrt{3}}{2} \sin^{-1} \frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{2} \left(\frac{x}{\sqrt{3}} \right) \left(\frac{\sqrt{3-x^2}}{\sqrt{3}} \right) + \text{constant}$$

$$= \frac{\sqrt{3}}{2} \sin^{-1} \frac{x}{\sqrt{3}} + \frac{x\sqrt{3(3-x^2)}}{6} + \text{constant}$$
1A

4. Let $11x = 3 \sin \theta$. Then $11 dx = 3 \cos \theta d\theta$. 1M

$$\int \sqrt{9 - 121x^2} dx = \frac{3}{11} \int \sqrt{9 - 9 \sin^2 \theta} \cos \theta d\theta$$

$$= \frac{9}{11} \int \cos^2 \theta d\theta$$
1A

$$= \frac{9}{22} \int (1 + \cos 2\theta) d\theta$$
1M

$$= \frac{9}{22} \theta + \frac{9}{44} \sin 2\theta + \text{constant}$$

$$= \frac{9}{22} \theta + \frac{9}{22} \sin \theta \cos \theta + \text{constant}$$

$$= \frac{9}{22} \sin^{-1} \frac{11x}{3} + \frac{9}{22} \left(\frac{11x}{3} \right) \left(\frac{\sqrt{9 - 121x^2}}{3} \right) + \text{constant}$$

$$= \frac{9}{22} \sin^{-1} \frac{11x}{3} + \frac{x\sqrt{9 - 121x^2}}{2} + \text{constant}$$
1A

5. Let $2\sqrt{2}x = \tan \theta$. Then $2\sqrt{2} dx = \sec^2 \theta d\theta$. 1M

$$\int \frac{dx}{(1 + 8x^2)^2} = \frac{1}{2\sqrt{2}} \int \frac{\sec^2 \theta}{(1 + \tan^2 \theta)^2} d\theta$$

$$= \frac{1}{2\sqrt{2}} \int \cos^2 \theta d\theta$$
1A

$$= \frac{1}{4\sqrt{2}} \int (1 + \cos 2\theta) d\theta$$
1M

$$= \frac{1}{4\sqrt{2}} \theta + \frac{1}{8\sqrt{2}} \sin 2\theta + \text{constant}$$

$$= \frac{1}{4\sqrt{2}} \theta + \frac{1}{4\sqrt{2}} \sin \theta \cos \theta + \text{constant}$$

$$= \frac{1}{4\sqrt{2}} \tan^{-1}(2\sqrt{2}x) + \frac{1}{4\sqrt{2}} \left(\frac{2\sqrt{2}x}{\sqrt{1 + 8x^2}} \right) \left(\frac{1}{\sqrt{1 + 8x^2}} \right) + \text{constant}$$

$$= \frac{1}{4\sqrt{2}} \tan^{-1}(2\sqrt{2}x) + \frac{x}{2(1 + 8x^2)} + \text{constant}$$
1A

6. Let $x = 7 \sin \theta$. Then $dx = 7 \cos \theta d\theta$. 1M

$$\begin{aligned} \int \frac{x^2}{\sqrt{49 - x^2}} dx &= \int \frac{49 \sin^2 \theta \cdot 7 \cos \theta}{\sqrt{49 - 49 \sin^2 \theta}} d\theta \\ &= 49 \int \sin^2 \theta d\theta \end{aligned} \quad 1A$$

$$= \frac{49}{2} \int (1 - \cos 2\theta) d\theta \quad 1M$$

$$= \frac{49\theta}{2} - \frac{49 \sin 2\theta}{4} + \text{constant}$$

$$= \frac{49\theta}{2} - \frac{49 \sin \theta \cos \theta}{2} + \text{constant}$$

$$= \frac{49}{2} \sin^{-1} \frac{x}{7} - \frac{49}{2} \left(\frac{x}{7} \right) \left(\frac{\sqrt{49 - x^2}}{7} \right) + \text{constant}$$

$$= \frac{49}{2} \sin^{-1} \frac{x}{7} - \frac{x\sqrt{49 - x^2}}{2} + \text{constant} \quad 1A$$

7. $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x} + 1}$

Let $e^x = \tan \theta$. Then $e^x dx = \sec^2 \theta d\theta$. 1M

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{\sec^2 \theta d\theta}{\tan^2 \theta + 1} \quad 1M$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \tan^{-1}(e^x) + C \quad 1A$$

8. (a) $x^2 + 2x \sin \frac{5\pi}{12} + 1 = x^2 + 2x \sin \frac{5\pi}{12} + \sin^2 \frac{5\pi}{12} + \cos^2 \frac{5\pi}{12}$

$$= \left(x + \sin \frac{5\pi}{12} \right)^2 + \cos^2 \frac{5\pi}{12} \quad 1A$$

(b) Let $x + \sin \frac{5\pi}{12} = \cos \frac{5\pi}{12} \tan \theta$. Then $dx = \cos \frac{5\pi}{12} \sec^2 \theta d\theta$. 1M

$$\begin{aligned} \int \frac{\cos \frac{5\pi}{12}}{x^2 + 2x \sin \frac{5\pi}{12} + 1} dx &= \int \frac{\cos \frac{5\pi}{12}}{\cos^2 \frac{5\pi}{12} \tan^2 \theta + \cos^2 \frac{5\pi}{12}} \cdot \cos \frac{5\pi}{12} \sec^2 \theta d\theta \\ &= \int d\theta \end{aligned} \quad 1A$$

$$= \theta + \text{constant}$$

$$= \tan^{-1} \left(\frac{x + \sin \frac{5\pi}{12}}{\cos \frac{5\pi}{12}} \right) + \text{constant} \quad 1A$$

$$= \tan^{-1} \left(\frac{x \csc \frac{5\pi}{12} + 1}{\cot \frac{5\pi}{12}} \right) + \text{constant}$$

$$= \tan^{-1} \frac{x(\sqrt{6} - \sqrt{2}) + 1}{2 - \sqrt{3}} + \text{constant} \quad 1A$$

9. $x^2 + 8x + 17 = (x + 4)^2 + 1$

Let $x + 4 = \tan \theta$. Then $dx = \sec^2 \theta d\theta$. 1M

$$\begin{aligned}\int \frac{x}{x^2 + 8x + 17} dx &= \int \frac{(\tan \theta - 4) \sec^2 \theta}{\tan^2 \theta + 1} d\theta \\ &= \int (\tan \theta - 4) d\theta\end{aligned}$$
1A

Let $u = \cos \theta$. Then $du = -\sin \theta d\theta$. 1M

$$\begin{aligned}\int \frac{x}{x^2 + 8x + 17} dx &= - \int \frac{du}{u} - \int 4 d\theta \\ &= -\ln |\cos \theta| - 4\theta + \text{constant} \\ &= -\ln \left| \frac{1}{\sqrt{x^2 + 8x + 17}} \right| - 4 \tan^{-1}(x + 4) + \text{constant} \\ &= \ln \sqrt{x^2 + 8x + 17} - 4 \tan^{-1}(x + 4) + \text{constant}\end{aligned}$$
1A

10. Let $x^2 = 4 \tan \theta$. Then $2x dx = 4 \sec^2 \theta d\theta$. 1M

$$\begin{aligned}\int \frac{x dx}{x^4 + 16} &= \frac{1}{2} \int \frac{4 \sec^2 \theta d\theta}{16 \tan^2 \theta + 16} \\ &= \frac{1}{8} \int d\theta \\ &= \frac{\theta}{8} + \text{constant} \\ &= \frac{1}{8} \tan^{-1} \frac{x^2}{4} + \text{constant}\end{aligned}$$
1A

11. (a) $8x - x^2 = -[x^2 - 2(4)(x) + 4^2] + 16$
 $= 16 - (x - 4)^2$

1A

(b) $\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$.

Let $x - 4 = 4 \sin \theta$. Then $dx = 4 \cos \theta d\theta$. 1M

$$\begin{aligned}\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{4 \cos \theta d\theta}{\sqrt{16 - 16 \sin^2 \theta}} \\ &= \int d\theta \\ &= \theta + \text{constant} \\ &= \sin^{-1} \frac{x - 4}{4} + \text{constant}\end{aligned}$$
1A