M2REG-SPVP-2223-ASM-SET 3-MATH
Suggested solutions

1. Required vector $= \left[\dfrac{(\mathbf{a}+\mathbf{b})\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right]\mathbf{a}$

$$= \frac{(5\mathbf{i}+\mathbf{j}-3\mathbf{k})\cdot(3\mathbf{i}-2\mathbf{j}+\mathbf{k})}{3^2+2^2+1^2}(3\mathbf{i}-2\mathbf{j}+\mathbf{k}) \qquad \text{1M}$$

$$= \frac{15}{7}\mathbf{i}-\frac{10}{7}\mathbf{j}+\frac{5}{7}\mathbf{k} \qquad \text{1A}$$

2. Required vector $= \dfrac{(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})}{(\mathbf{a}-2\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})}(\mathbf{a}-2\mathbf{b})$

$$= \frac{(-3\mathbf{i}+7\mathbf{j}-\mathbf{k})\cdot(3\mathbf{i}+\mathbf{j}-10\mathbf{k})}{3^2+1^2+10^2}(3\mathbf{i}+\mathbf{j}-10\mathbf{k}) \qquad \text{1M}$$

$$= \frac{12}{55}\mathbf{i}+\frac{4}{55}\mathbf{j}-\frac{8}{11}\mathbf{k} \qquad \text{1A}$$

3. Projection vector $= \dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}(\mathbf{v})$

$$= \frac{[3\mathbf{i}+\lambda\mathbf{j}+(\lambda+1)\mathbf{k}]\cdot(3\mathbf{i}+2\mathbf{j}-\mathbf{k})}{3^2+2^2+1^2}(3\mathbf{i}+2\mathbf{j}-\mathbf{k}) \qquad \text{1M}$$

$$= \frac{3\lambda+24}{14}\mathbf{i}+\frac{\lambda+8}{7}\mathbf{j}-\frac{\lambda+8}{14}\mathbf{k}$$

We have $\dfrac{3\lambda+24}{14}=\mu$, $\dfrac{\lambda+8}{7}=\mu-\dfrac{1}{2}$ and $\beta=-\dfrac{\lambda+8}{14}$. 1M

Solving, we have $\lambda=-1$, $\mu=\dfrac{3}{2}$ and $\beta=-\dfrac{1}{2}$. 3A

4. Projection vector $= \dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}(\mathbf{v})$

$$= \frac{[(2\lambda-1)\mathbf{i}+3\mathbf{j}-5\mathbf{k}]\cdot(2\mathbf{i}-\lambda\mathbf{j}-\mathbf{k})}{2^2+\lambda^2+1^2}(2\mathbf{i}-\lambda\mathbf{j}-\mathbf{k}) \qquad \text{1M}$$

$$= \frac{2\lambda+6}{\lambda^2+5}\mathbf{i}-\frac{\lambda^2+3\lambda}{\lambda^2+5}\mathbf{j}-\frac{\lambda+3}{\lambda^2+5}\mathbf{k}$$

We have $\dfrac{2\lambda+6}{\lambda^2+5}=\dfrac{4}{3}$, $-\dfrac{\lambda+3}{\lambda^2+5}=-\dfrac{2}{3}$ and $-\dfrac{\lambda^2+3\lambda}{\lambda^2+5}=\mu$.

$$\frac{2\lambda+6}{\lambda^2+5}=\frac{4}{3} \quad \text{and} \quad -\frac{\lambda+3}{\lambda^2+5}=-\frac{2}{3} \qquad \text{1M}$$

$$6\lambda+18=4\lambda^2+20 \qquad 3\lambda+9=2\lambda^2+10$$

$$4\lambda^2-6\lambda+2=0 \qquad 2\lambda^2-3\lambda+1=0$$

$$\lambda=1 \text{ or } \frac{1}{2} \qquad \lambda=1 \text{ or } \frac{1}{2}$$

When $\lambda=1$, $\mu=-\dfrac{2}{3}$; when $\lambda=\dfrac{1}{2}$, $\mu=-\dfrac{1}{3}$.

Thus, $(\lambda,\ \mu)=\left(1,\ -\dfrac{2}{3}\right)$ or $\left(\dfrac{1}{2},\ -\dfrac{1}{3}\right)$. 1A+1A

5. (a) $\overrightarrow{AB} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$\overrightarrow{AC} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

$$\text{Required vector} = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{(AC)^2}(\overrightarrow{AC})$$

$$= \frac{(\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})}{2^2 + 2^2 + 2^2}(2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$ 1M

$$= \frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$ 1A

(b) Let N be the projection of B onto AC.

$\overrightarrow{BN} = \overrightarrow{AN} - \overrightarrow{AB}$

$$= -\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{4}{3}\mathbf{k}$$ 1A

Required distance $= BN$

$$= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2}$$

$$= \frac{2\sqrt{6}}{3}$$ 1A

6. (a) $\overrightarrow{OA} = -\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

$\overrightarrow{OB} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{(OB)^2}(\overrightarrow{OB})$$

$$= \frac{(-\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})}{1^2 + 2^2 + 2^2}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$ 1M

$$= -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$ 1A

(b) $$OM = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$$

$$= 1$$ 1A

$$\overrightarrow{AM} = \frac{2}{3}\mathbf{i} - \frac{10}{3}\mathbf{j} - \frac{11}{3}\mathbf{k}$$

$$AM = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{10}{3}\right)^2 + \left(\frac{11}{3}\right)^2}$$

$$= 5$$

$$\text{Required area} = \frac{1}{2}(OM)(AM)$$

$$= \frac{5}{2}$$ 1A

7. (a) $\overrightarrow{AB} = -3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

$\overrightarrow{AC} = \mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & -2 \\ 1 & -5 & -4 \end{vmatrix}$$

$$= -2\mathbf{i} - 14\mathbf{j} + 17\mathbf{k}$$ 1A

Required area $= \dfrac{1}{2}\sqrt{2^2 + 14^2 + 17^2}$

$= \dfrac{\sqrt{489}}{2}$ 1A

(b) $\overrightarrow{AD} = \mathbf{i} + \mathbf{k}$

Required magnitude $= \dfrac{|(\mathbf{i} + \mathbf{k}) \cdot (-2\mathbf{i} - 14\mathbf{j} + 17\mathbf{k})|}{\sqrt{2^2 + 14^2 + 17^2}}$ 1M

$= \dfrac{5\sqrt{489}}{163}$ 1A

(c) Required volume $= \dfrac{1}{3}\left(\dfrac{\sqrt{489}}{2}\right)\left(\dfrac{5\sqrt{489}}{163}\right)$ 1M

$= \dfrac{5}{2}$ 1A

8. (a) $\overrightarrow{PA} = (6-t)\mathbf{i} - 12\mathbf{k}$ and $\overrightarrow{PB} = (1-t)\mathbf{i} + 5\mathbf{j} - 12\mathbf{k}$. 1M

$$\sqrt{(6-t)^2 + 12^2} = \sqrt{(1-t)^2 + 5^2 + 12^2} \quad \text{1M}$$

$$t^2 - 12t + 36 = t^2 - 2t + 26$$

$$t = 1$$

(b) (i) $\overrightarrow{CA} = -\mathbf{i} - 8\mathbf{j}$

$\overrightarrow{CB} = -6\mathbf{i} - 3\mathbf{j}$

$$\overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -8 & 0 \\ -6 & -3 & 0 \end{vmatrix} \quad \text{1M}$$

$$= -45\mathbf{k}$$

Required unit vectors are $\mathbf{k}$ and $-\mathbf{k}$. 1A

(ii) $\overrightarrow{CD} = -3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

Let θ be the required angle.

$$\overrightarrow{CD} \cdot \mathbf{k} = (CD)|\mathbf{k}|\cos\theta \quad \text{1M}$$

$$5 = \sqrt{3^2 + 4^2 + 5^2}\cos\theta$$

$$\theta = 45^\circ \quad \text{1A}$$

(iii) $$\overrightarrow{DE} = \frac{\overrightarrow{DC} \cdot \mathbf{k}}{1^2}(\mathbf{k}) \quad \text{1M}$$

$$= -5\mathbf{k}$$

$$\overrightarrow{PE} = \overrightarrow{DE} + \overrightarrow{OD} - \overrightarrow{OP}$$

$$= 3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$$

$$PE = \sqrt{3^2 + 4^2 + 12^2}$$

$$= 13 \quad \text{1A}$$

Since $PA = PB = PE = 13$, E is on the sphere.

The claim is agreed. 1A

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