

M2REG-AOD-2223-ASM-SET 5-MATH

Suggested solutions

1. (a) $\tan \theta = \tan(\angle APO - \angle BPO)$

$$= \frac{\tan \angle APO - \tan \angle BPO}{1 + \tan \angle APO \tan \angle BPO}$$

1M

$$= \frac{\frac{16}{y} - \frac{10}{y}}{1 + \frac{16}{y} \cdot \frac{10}{y}}$$

$$= \frac{6y}{y^2 + 160}$$

1A

(b) $\tan \theta = \frac{6y}{y^2 + 160}$

$$\sec^2 \theta \frac{d\theta}{dy} = \frac{6(y^2 + 160) - 6y(2y)}{(y^2 + 160)^2}$$

1M

$$\frac{d\theta}{dy} = \frac{6(160 - y^2) \cos^2 \theta}{(y^2 + 160)^2}$$

1A

When $\frac{d\theta}{dy} = 0$, $y = \sqrt{160}$.

1A

y	$0 < y < \sqrt{160}$	$y > \sqrt{160}$
$\frac{d\theta}{dy}$	+	-

1M

θ is maximum at $y = \sqrt{160}$.

$$\text{Required value} = \tan^{-1} \frac{6\sqrt{160}}{(\sqrt{160})^2 + 160}$$

$$\approx 0.233 \text{ rad}$$

1A

2. $h^2 + r^2 = 4^2$ 1M

$$r^2 = 16 - h^2$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(16 - h^2)h$$
 1M

$$\frac{dV}{dh} = \frac{1}{3}\pi(16 - 3h^2)$$
 1M

When $\frac{dV}{dh} = 0$, $h = \frac{4}{\sqrt{3}}$ or $-\frac{4}{\sqrt{3}}$ (rejected).

h	$0 < h < \frac{4}{\sqrt{3}}$	$\frac{4}{\sqrt{3}} < h < 4$
$\frac{dV}{dh}$	+	-

1M

V is maximum at $h = \frac{4}{\sqrt{3}}$.

$$\begin{aligned} \text{Maximum volume} &= \frac{1}{3}\pi \left[16 \left(\frac{4}{\sqrt{3}} \right)^2 - \left(\frac{4}{\sqrt{3}} \right)^3 \right] \\ &= \frac{128\sqrt{3}\pi}{27} \end{aligned}$$
 1A

3. Let the x -coordinate of P be p , and the area of $\triangle MNP$ be A .

$$A = \frac{1}{2}(p) \left(3\sqrt{1 - \frac{p^2}{4}} \right)$$
 1M

$$= \frac{3p}{4} \sqrt{4 - p^2}$$

$$\frac{dA}{dp} = \frac{3}{4}(4 - p^2)^{\frac{1}{2}} + \frac{3p}{8}(4 - p^2)^{-\frac{1}{2}}(-2p)$$
 1M

$$= \frac{3}{2}(4 - p^2)^{-\frac{1}{2}}(2 - p^2)$$

When $\frac{dA}{dp} = 0$, $p = \sqrt{2}$. 1A

p	$0 < p < \sqrt{2}$	$\sqrt{2} < p < 2$
$\frac{dA}{dp}$	+	-

1M

A is maximum at $p = \sqrt{2}$.

$$\begin{aligned} \text{Maximum area} &= \frac{3\sqrt{2}}{4} \sqrt{4 - (\sqrt{2})^2} \\ &= \frac{3}{2} \end{aligned}$$
 1A

4. (a) Let T hours be the time taken.

$$T = \frac{x}{8} + \frac{\sqrt{(6-x)^2 + 4^2}}{3} \quad 1A$$

$$= \frac{x}{8} + \frac{\sqrt{x^2 - 12x + 52}}{3}$$

(b) $\frac{dT}{dx} = \frac{1}{8} + \frac{1}{6}(x^2 - 12x + 52)^{-\frac{1}{2}}(2x - 12) \quad 1M$

$$= \frac{1}{8} + \frac{2x - 12}{6(x^2 - 12x + 52)^{\frac{1}{2}}}$$

When $\frac{dT}{dx} = 0$,

$$\frac{1}{8} + \frac{2x - 12}{6\sqrt{x^2 - 12x + 52}} = 0 \quad 1M$$

$$(8x - 48)^2 = 3^2(x^2 - 12x + 52)$$

$$55x^2 - 660x + 1836 = 0$$

$$x = \frac{330 - 12\sqrt{55}}{55} \quad \text{or} \quad \frac{330 + 12\sqrt{55}}{55} \text{ (rejected)} \quad 1A$$

x	$0 < x < \frac{330 - 12\sqrt{55}}{55}$	$\frac{330 - 12\sqrt{55}}{55} < x < 6$
$\frac{dT}{dx}$	-	+

1M

T attains minimum when $x = \frac{330 - 12\sqrt{55}}{55} \approx 4.38$.

Minimum time taken = $\frac{x}{8} + \frac{\sqrt{x^2 - 12x + 52}}{3}$

$$\approx 1.99 \text{ h} > 1.5 \text{ h}$$

1A

The claim is disagreed.

1A

$$5. \quad (a) \quad HB = \sqrt{x^2 + (\sqrt{3})^2} \quad 1M$$

$$= \sqrt{x^2 + 3} \text{ m}$$

$$\frac{BC}{BK} = \frac{AB}{HB}$$

$$\frac{x+y}{x} = \frac{9}{\sqrt{x^2+3}} \quad 1M$$

$$y = \frac{9x}{\sqrt{x^2+3}} - x \quad 1$$

$$(b) \quad \frac{dy}{dx} = \frac{9(x^2+3)^{\frac{1}{2}} - \frac{9x}{2}(x^2+3)^{-\frac{1}{2}}(2x)}{x^2+3} - 1 \quad 1M$$

$$= \frac{9[(x^2+3) - x^2]}{(x^2+3)^{\frac{3}{2}}} - 1$$

$$= \frac{27}{(x^2+3)^{\frac{3}{2}}} - 1$$

$$\text{When } \frac{dy}{dx} = 0,$$

$$(x^2+3)^{\frac{3}{2}} = 27$$

$$x^2+3 = 9$$

$$x = \pm\sqrt{6}$$

x	$0 < x < \sqrt{6}$	$\sqrt{6} < x < \sqrt{78}$
x	+	-

1M

y attains its maximum at $x = \sqrt{6}$.

$$\begin{aligned} \text{Required value} &= \frac{9(\sqrt{6})}{\sqrt{(\sqrt{6})^2+3}} - \sqrt{6} \\ &= 2\sqrt{6} \end{aligned}$$

1A

$$(c) \quad (i) \quad \frac{AC}{HK} = \frac{AB}{HB}$$

$$AC = \frac{9\sqrt{3}}{\sqrt{x^2+3}}$$

$$S = \frac{\left(\frac{9\sqrt{3}}{\sqrt{x^2+3}} + \sqrt{3}\right)\left(\frac{9x}{\sqrt{x^2+3}} - x\right)}{2}$$

$$= \frac{81\sqrt{3}x}{2(x^2+3)} - \frac{\sqrt{3}x}{2}$$

1

$$(ii) \quad \frac{dS}{dx} = \frac{81\sqrt{3}}{2} \left[\frac{(x^2+3) - x(2x)}{(x^2+3)^2} \right] - \frac{\sqrt{3}}{2} \quad 1M$$

$$= \frac{81\sqrt{3}(3-x^2)}{2(x^2+3)^2} - \frac{\sqrt{3}}{2}$$

$$\left. \frac{dS}{dx} \right|_{x=\sqrt{6}} = \frac{81\sqrt{3}[3-(\sqrt{6})^2]}{2[(\sqrt{6})^2+3]^2} - \frac{\sqrt{3}}{2}$$

$$= -2\sqrt{3} \neq 0$$

1M

Thus, S is not a maximum when y attains its maximum.

1A

6. (a) $100e^{-2.5} = [-(25)^2 + A(25) + B]e^{-\frac{25}{10}}$ 1M

$$25A + B = 725$$

$$75e^{-3} = [-(30)^2 + A(30) + B]e^{-\frac{30}{10}}$$

$$30A + B = 975$$

Solving, we have $A = 50$ and $B = -525$.

1A+1A

(b) (i) $\frac{dW(x)}{dx} = (-x^2 + 50x - 525)e^{-\frac{x}{10}} \left(-\frac{1}{10}\right) + e^{-\frac{x}{10}}(-2x + 50)$ 1M

$$= \frac{1}{10}(x^2 - 70x + 1025)e^{-\frac{x}{10}}$$
1A

(ii) When $\frac{dW(x)}{dx} = 0$,

$$x^2 - 70x + 1025 = 0$$
1M

$$x = 35 - 10\sqrt{2} \quad \text{or} \quad 35 + 10\sqrt{2} \text{ (rejected)}$$
1A

x	$15 < x < 35 - 10\sqrt{2}$	$35 - 10\sqrt{2} < x < 35$
$\frac{dW(x)}{dx}$	+	-

1M

$W(x)$ attains its maximum at $x = 35 - 10\sqrt{2}$.

$$\text{Maximum weight} = [-(35 - 10\sqrt{2})^2 + 50(35 - 10\sqrt{2}) - 525]e^{-\frac{35-10\sqrt{2}}{10}}$$

$$\approx 10.29 \text{ t}$$

1A

(c) $P(x) = \frac{1000W(x)}{x}$

$$\frac{dP(x)}{dx} = 1000 \left[\frac{xW'(x) - W(x)}{x^2} \right]$$
1M

$$\left. \frac{dP(x)}{dx} \right|_{x=35-10\sqrt{2}} = 1000 \left[\frac{(35 - 10\sqrt{2})W'(35 - 10\sqrt{2}) - W(35 - 10\sqrt{2})}{(35 - 10\sqrt{2})^2} \right]$$

$$= 1000 \left[\frac{-W(35 - 10\sqrt{2})}{(35 - 10\sqrt{2})^2} \right]$$

$$\neq 0$$

1M

$P(x)$ does not attain its maximum when $W(x)$ attains its maximum.

The claim is incorrect.

1A