M2INT-INT-2223-ASM-SET 1-MATH
Suggested solutions

1. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos 2x}{\cos x + \sin x}\,\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x}\,\mathrm{d}x$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x)\,\mathrm{d}x$$ 1M

$$= \Big[\sin x + \cos x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$ 1A

$$= (1 + 0) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$$

$$= 1 - \sqrt{2}$$ 1A

2. $$\int_0^{\frac{\pi}{4}} \sin^4 x\,\mathrm{d}x = \frac{1}{4}\int_0^{\frac{\pi}{4}} (1 - \cos 2x)^2\,\mathrm{d}x$$ 1M

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} (1 - 2\cos 2x + \cos^2 2x)\,\mathrm{d}x$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right]\mathrm{d}x$$ 1M

$$= \frac{1}{4}\left[\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8}\right]_0^{\frac{\pi}{4}}$$ 1A

$$= \frac{1}{4}\left(\frac{3\pi}{8} - 1 + 0\right) - \frac{1}{4}(0)$$

$$= \frac{3\pi}{32} - \frac{1}{4}$$ 1A

3. (a) Let $u = 1 - e^{-x}$. Then $\mathrm{d}u = e^{-x}\,\mathrm{d}x$. 1M

When $x = \ln 2$, $u = \dfrac{1}{2}$; when $x = \ln 3$, $u = \dfrac{2}{3}$.

$$\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}}\,\mathrm{d}x = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{\mathrm{d}u}{u} \quad \text{1M}$$

$$= \Big[\ln|u|\Big]_{\frac{1}{2}}^{\frac{2}{3}} \quad \text{1A}$$

$$= \ln\frac{2}{3} - \ln\frac{1}{2}$$

$$= \ln\frac{4}{3} \quad \text{1A}$$

(b) $$\frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1) + Bu}{u(u-1)}$$

$$= \frac{(A+B)u - A}{u(u-1)}$$

So, $A + B = 0$ and $-A = 1$. 1M

Solving, we have $A = -1$ and $B = 1$. 1A

(c) $$\int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x - 1)}\,\mathrm{d}x = \int_{\ln 2}^{\ln 3} \left(\frac{1}{e^x - 1} - \frac{1}{e^x}\right)\mathrm{d}x \quad \text{1M}$$

$$= \int_{\ln 2}^{\ln 3} \left(\frac{e^{-x}}{1 - e^{-x}} - e^{-x}\right)\mathrm{d}x$$

$$= \ln\frac{4}{3} + \Big[e^{-x}\Big]_{\ln 2}^{\ln 3} \quad \text{1M+1A}$$

$$= \ln\frac{4}{3} + \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \ln\frac{4}{3} - \frac{1}{6} \quad \text{1A}$$

4. Let $u = \dfrac{3}{2} - \dfrac{2}{x}$. Then $\mathrm{d}u = \dfrac{2}{x^2}\,\mathrm{d}x$. 1M

$$\int \frac{\mathrm{d}x}{x^2\sqrt{\frac{3}{2} - \frac{2}{x}}} = \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u}} \quad \text{1M}$$

$$= u^{\frac{1}{2}} + C$$

$$= \left(\frac{3}{2} - \frac{2}{x}\right)^{\frac{1}{2}} + C \quad \text{1A}$$

5. Let $u = \frac{1}{x}$. Then $\mathrm{d}u = -\frac{1}{x^2}\,\mathrm{d}x$

$$\int \frac{\sqrt[3]{4x^3 - 2}}{x^5}\,\mathrm{d}x = \int \sqrt[3]{4 - \frac{2}{x^3}} \cdot \frac{1}{x^4}\,\mathrm{d}x \qquad \text{1M}$$

$$= -\int \sqrt[3]{4 - 2u^3}\,u^2\,\mathrm{d}u \qquad \text{1M}$$

Let $w = 4 - 2u^3$. Then $\mathrm{d}w = -6u^2\,\mathrm{d}u$. 1M

$$\int \frac{\sqrt[3]{4x^3 - 2}}{x^5}\,\mathrm{d}x = \frac{1}{6}\int \sqrt[3]{w}\,\mathrm{d}w \qquad \text{1M}$$

$$= \frac{1}{8}w^{\frac{4}{3}} + \text{constant}$$

$$= \frac{1}{8}\left(4 - \frac{2}{x^3}\right)^{\frac{4}{3}} + \text{constant} \qquad \text{1A}$$

6. Let $u = 1 - 2x^2$. Then $\mathrm{d}u = -4x\,\mathrm{d}x$. 1M

$$\int \frac{2x^3\,\mathrm{d}x}{(1 - 2x^2)^{\frac{1}{3}}} = -\frac{1}{4}\int \frac{1 - u}{u^{\frac{1}{3}}}\,\mathrm{d}u \qquad \text{1M}$$

$$= -\frac{1}{4}\int \left(u^{-\frac{1}{3}} - u^{\frac{2}{3}}\right)\mathrm{d}u \qquad \text{1M}$$

$$= -\frac{3}{8}u^{\frac{2}{3}} + \frac{3}{20}u^{\frac{5}{3}} + \text{constant}$$

$$= -\frac{3}{8}(1 - 2x^2)^{\frac{2}{3}} + \frac{3}{20}(1 - 2x^2)^{\frac{5}{3}} + \text{constant} \qquad \text{1A}$$

7. (a) Let $u = x + 2$. Then $\mathrm{d}u = \mathrm{d}x$. 1M

$$\int \frac{x - 1}{x + 2}\,\mathrm{d}x = \int \frac{u - 3}{u}\,\mathrm{d}u$$

$$= \int \left(1 - \frac{3}{u}\right)\mathrm{d}u$$

$$= u - 3\ln|u| + \text{constant}$$

$$= x - 3\ln|x + 2| + \text{constant} \qquad \text{1A}$$

(b) Let $u = \sqrt{x}$. Then $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$. 1M

$$\int \frac{\sqrt{x} - 1}{x + 2\sqrt{x}}\,\mathrm{d}x = 2\int \left(\frac{\sqrt{x} - 1}{\sqrt{x} + 2} \cdot \frac{1}{2\sqrt{x}}\right)\mathrm{d}x$$

$$= 2\int \frac{u - 1}{u + 2}\,\mathrm{d}u \qquad \text{1A}$$

$$= 2u - 6\ln|u + 2| + \text{constant} \qquad \text{1A}$$

$$= 2\sqrt{x} - 6\ln\left(\sqrt{x} + 2\right) + \text{constant} \qquad \text{1A}$$

8. (a) Let $x = 3\tan\theta$. Then $\mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta$. 1M

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \tan^{-1}\dfrac{1}{3}$.

$$\int_0^1 \frac{\mathrm{d}x}{x^2+9} = \int_0^{\tan^{-1}\frac{1}{3}} \frac{3\sec^2\theta\,\mathrm{d}\theta}{9\tan^2\theta+9} \quad \text{1M}$$

$$= \frac{1}{3}\int_0^{\tan^{-1}\frac{1}{3}} \mathrm{d}\theta$$

$$= \frac{1}{3}\Big[\theta\Big]_0^{\tan^{-1}\frac{1}{3}} \quad \text{1A}$$

$$= \frac{1}{3}\tan^{-1}\frac{1}{3} \quad \text{1A}$$

(b) $$\int_0^1 \frac{x^2}{x^2+9}\,\mathrm{d}x = \int_0^1 \left(1 - \frac{9}{x^2+9}\right)\mathrm{d}x \quad \text{1M}$$

$$= \Big[x\Big]_0^1 - 9\times\frac{1}{3}\tan^{-1}\frac{1}{3}$$

$$= 1 - 3\tan^{-1}\frac{1}{3} \quad \text{1A}$$

9. Let $x = \sqrt{3}\tan\theta$. Then $\mathrm{d}x = \sqrt{3}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_{-3}^{3} \frac{1}{(x^2+3)^{\frac{3}{2}}}\,\mathrm{d}x = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3}\sec^2\theta}{(3\tan^2\theta+3)^{\frac{3}{2}}}\,\mathrm{d}\theta \quad \text{1A}$$

$$= \frac{1}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\sec^3\theta}\,\mathrm{d}\theta$$

$$= \frac{1}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\theta\,\mathrm{d}\theta$$

$$= \frac{1}{3}\Big[\sin\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{3} \quad \text{1A}$$

10. Let $x = \sin^2\theta$, where $0 \le \theta \le \dfrac{\pi}{2}$. Then $\mathrm{d}x = 2\sin\theta\cos\theta\,\mathrm{d}x$.

When $x = 0$, $\theta = 0$; when $x = \dfrac{3}{4}$, $\theta = \dfrac{\pi}{3}$. 1A

$$\int_0^{\frac{3}{4}} \sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\cdot 2\sin\theta\cos\theta\,\mathrm{d}\theta \quad \text{1M}$$

$$= \int_0^{\frac{\pi}{3}} 2\sin^2\theta\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{3}} (1-\cos 2\theta)\,\mathrm{d}\theta \quad \text{1M}$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}} \quad \text{1A}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4} \quad \text{1A}$$

11. (a) $\displaystyle\int_{-1}^{1} \frac{\mathrm{d}x}{x^2+4x+7} = \int_{-1}^{1} \frac{\mathrm{d}x}{(x+2)^2+3}$ 1M

Let $x+2 = \sqrt{3}\tan\theta$. Then $\mathrm{d}x = \sqrt{3}\sec^2\theta \,\mathrm{d}\theta$. 1M

When $x=-1$, $\theta = \dfrac{\pi}{6}$; when $x=1$, $\theta = \dfrac{\pi}{3}$.

$$\int_{-1}^{1} \frac{\mathrm{d}x}{x^2+4x+7} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{3}\sec^2\theta}{3\sec^2\theta} \qquad \text{1M}$$

$$= \frac{1}{\sqrt{3}}\Big[\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \qquad \text{1A}$$

$$= \frac{\sqrt{3}\pi}{18} \qquad \text{1A}$$

(b) Let $u = x^2+4x+7$. Then $\mathrm{d}x = (2x+4)\,\mathrm{d}x$. 1M

When $x=-1$, $u=4$; when $x=1$, $u=12$.

$$\int_{-1}^{1} \frac{x}{x^2+4x+7}\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} \frac{2x+4}{x^2+4x+7}\,\mathrm{d}x - \int_{-1}^{1} \frac{2}{x^2+4x+7}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{2}\int_{4}^{12} \frac{\mathrm{d}u}{u} - 2\times\frac{\sqrt{3}\pi}{18} \qquad \text{1M}$$

$$= \frac{1}{2}\Big[\ln|u|\Big]_{4}^{12} - \frac{\sqrt{3}\pi}{9} \qquad \text{1A}$$

$$= \frac{1}{2}\ln 3 - \frac{\sqrt{3}\pi}{9} \qquad \text{1A}$$

12. (a) Let $u = e^{3x}+3x+1$. Then $\mathrm{d}u = 3(e^{3x}+1)\,\mathrm{d}x$. 1M

When $x=1$, $u = e^3+4$; when $x=4$, $u = e^{12}+13$.

$$\int_{1}^{4} \frac{e^{3x}+1}{e^{3x}+3x+1}\,\mathrm{d}x = \frac{1}{3}\int_{e^3+4}^{e^{12}+13} \frac{\mathrm{d}u}{u} \qquad \text{1M}$$

$$= \frac{1}{3}\Big[\ln|u|\Big]_{e^3+4}^{e^{12}+13} \qquad \text{1A}$$

$$= \frac{1}{3}\ln\frac{e^{12}+13}{e^3+4} \qquad \text{1A}$$

(b) $$\int_{1}^{4} \frac{x\,\mathrm{d}x}{e^{3x}+3x+1} = \frac{1}{3}\int_{1}^{4} \frac{e^{3x}+3x+1-(e^{3x}+1)}{e^{3x}+3x+1}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{3}\Big[x\Big]_{1}^{4} - \frac{1}{3}\times\frac{1}{3}\ln\frac{e^{12}+13}{e^3+4} \qquad \text{1M}$$

$$= 1 - \frac{1}{9}\ln\frac{e^{12}+13}{e^3+4} \qquad \text{1A}$$

13. (a) Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \dfrac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} \qquad \text{1A}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + 0 - 0 - 0\right)$$

$$= \frac{\pi}{4} \qquad \text{1A}$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)^{\frac{3}{2}}x] = (1-x^2)^{\frac{3}{2}} + \frac{3}{2}(1-x^2)^{\frac{1}{2}}(-2x)(x) \qquad \text{1M}$$

$$= \sqrt{1-x^2}[(1-x^2) - 3x^2]$$

$$= \sqrt{1-x^2} - 4x^2\sqrt{1-x^2} \qquad \text{1}$$

(c) By (b), $4x^2\sqrt{1-x^2} = \sqrt{1-x^2} - \dfrac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)^{\frac{3}{2}}x]$

$$\int_0^1 x^2\sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{4}\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x - \frac{1}{4}\left[(1-x^2)^{\frac{3}{2}}x\right]_0^1 \qquad \text{1M}$$

$$= \frac{1}{4}\times\frac{\pi}{4} - \frac{1}{4}(0-0)$$

$$= \frac{\pi}{16} \qquad \text{1A}$$

END OF PAPER