

REG-2223-MOCK-SET 2-MATH-EP(M2)

建議題解

1.	$\begin{aligned} \frac{d}{dx}(\cot 5x) &= \lim_{h \rightarrow 0} \frac{\cot(5x+5h) - \cot 5x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\tan 5x - \tan(5x+5h)}{h \tan 5x \tan(5x+5h)} \\ &= \lim_{h \rightarrow 0} \frac{\tan[5x - (5x+5h)][1 + \tan 5x \tan(5x+5h)]}{h \tan 5x \tan(5x+5h)} \\ &= \lim_{h \rightarrow 0} \frac{\tan(-5h) - 5[1 + \tan 5x \tan(5x+5h)]}{-5h \tan 5x \tan(5x+5h)} \\ &= \frac{-5(1 + \tan^2 5x)}{\tan^2 5x} \\ &= -5 \csc^2 5x \end{aligned}$	1M 1M 1M 1A
2. (a)	$\begin{aligned} (1+x)^2(4+ax)^5 &= (1+2x+x^2)(4^5 + C_1^5 4^4 ax + C_2^5 4^3 a^2 x^2 + C_3^5 4^2 a^3 x^3 + \dots) \\ &= (1+2x+x^2)(1024 + 1280ax + 640a^2 x^2 + 160a^3 x^3 + \dots) \\ 1(160a^3) + 2(640a^2) + 1(1280a) &= 3360 \\ a^3 + 8a^2 + 8a - 21 &= 0 \\ (a+3)(a^2 + 5a - 7) &= 0 \\ a = -3 \quad \text{或} \quad a^2 + 5a - 7 &= 0 \\ a = -3 \quad \text{或} \quad -6.14 \text{ (捨去)} \quad \text{或} \quad 1.14 \text{ (捨去)} \end{aligned}$	1M 1M 1A
(b)	$\begin{aligned} x^2 \text{ 的係數} &= 1(640a^2) + 2(1280a) + 1(1024) \\ &= -896 \end{aligned}$	1M 1A
3. (a)	$\begin{aligned} \int x(\ln x)^2 dx &= \frac{x^2}{2}(\ln x)^2 - \int \left(\frac{x^2}{2}\right)(2) \ln x \left(\frac{1}{x}\right) dx \\ &= \frac{x^2(\ln x)^2}{2} - \int x \ln x dx \\ &= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{1}{2} \int x^2 \times \frac{1}{x} dx \\ &= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + \text{常數} \end{aligned}$	1M+1A 1M 1A
(b)	$\begin{aligned} \text{體積} &= \pi \int_1^e x(\ln x)^2 dx \\ &= \pi \left[\frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} \right]_1^e \\ &= \frac{\pi(e^2 - 1)}{4} \end{aligned}$	1M 1A

解	分
4. (a) $\triangle OPQ$ 的面積 $= \frac{1}{2}(u)(ue^{2u}) = \frac{1}{2}u^2e^{2u}$。 (b) 設 v 為 P 的 y 坐標。 $v = ue^{2u}$ $\frac{dv}{dt} = e^{2u} \frac{du}{dt} + 2ue^{2u} \frac{du}{dt}$ 當 $u = 1$, $-3 = (e^2 + 2e^2) \frac{du}{dt}$ $\frac{du}{dt} = -\frac{1}{e^2}$ 設 A 為 $\triangle OPQ$ 的面積。 $A = \frac{1}{2}u^2e^{2u}$ $\frac{dA}{dt} = (u^2e^{2u} + ue^{2u}) \frac{du}{dt}$ 當 $u = 1$, $\begin{aligned} \frac{dA}{dt} &= (e^2 + e^2) \frac{du}{dt} \\ &= 2e^2 \left(-\frac{1}{e^2} \right) \\ &= -2 \end{aligned}$ 所求之率為 $-2/s$。	1A 1M 1M+1M 1A

解	分
5. (a) 當 $n = 1$, 左式 = $\sin \theta$, 右式 = $\frac{\sin^2 \theta}{\sin \theta} = \sin \theta =$ 左式 對 $n = 1$, 命題為真。 假設 $\sum_{k=1}^p \sin(2k-1)\theta = \frac{\sin^2 p\theta}{\sin \theta}$, 其中 $p \in \mathbb{Z}^+$ 。 $\begin{aligned} \sum_{k=1}^{p+1} \sin(2k-1)\theta &= \sum_{k=1}^p \sin(2k-1)\theta + \sin(2p+1)\theta \\ &= \frac{\sin^2 p\theta}{\sin \theta} + \sin(2p+1)\theta \\ &= \frac{1}{\sin \theta} [\sin^2 p\theta + \sin(2p+1)\theta \sin \theta] \\ &= \frac{1}{\sin \theta} \left(\frac{1 - \cos(2p\theta)}{2} + \frac{\cos(2p\theta) - \cos(2p+2)\theta}{2} \right) \\ &= \frac{1}{\sin \theta} \times \frac{1 - \cos(2p+2)\theta}{2} \\ &= \frac{\sin^2(p+1)\theta}{\sin \theta} \end{aligned}$ 對 $n = p+1$, 命題為真。 藉數學歸納法, $\forall n \in \mathbb{Z}^+$, 命題為真。 (b) $\sum_{k=135}^{234} \sin \frac{(2k-1)\pi}{9} = \sin \phi$ $\sum_{k=1}^{234} \sin \frac{(2k-1)\pi}{9} - \sum_{k=1}^{134} \sin \frac{(2k-1)\pi}{9} = \sin \phi$ $\frac{\sin^2 \frac{234\pi}{9}}{\sin \frac{\pi}{9}} - \frac{\sin^2 \frac{134\pi}{9}}{\sin \frac{\pi}{9}} = \sin \phi$ $0 - \frac{\sin^2 (15\pi - \frac{\pi}{9})}{\sin \frac{\pi}{9}} = \sin \phi$ $\sin \phi = -\sin \frac{\pi}{9}$ $\phi = -\frac{\pi}{9}$	1 1 1 1M 1 1M 1A 1A

6. (a) 設 $x = a \tan \theta$ 。則 $dx = a \sec^2 \theta d\theta$ 。

$$\begin{aligned}\int \frac{1}{x^2 + a^2} dx &= \int \frac{a \sec^2 \theta}{a^2 \tan^2 \theta + a^2} d\theta \\ &= \frac{1}{a} \theta + \text{常數} \\ &= \frac{1}{a} \tan^{-1} \frac{x}{a} + \text{常數}\end{aligned}$$

1M+1A

1

- (b) (i) $t = \tan \frac{\theta}{2}$

$$1 = \sec^2 \frac{\theta}{2} \times \frac{1}{2} \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{2}{1 + \tan^2 \frac{\theta}{2}}$$

$$= \frac{2}{1 + t^2}$$

1A

1

- (ii) 設 $t = \tan \frac{\theta}{2}$ 。則 $d\theta = \frac{2}{1 + t^2} dt$ 。

$$\cos \theta = 2 \cos^2 \frac{\theta}{2} - 1 = \frac{2}{1 + t^2} - 1 = \frac{1 - t^2}{1 + t^2}。$$

$$\begin{aligned}\int \frac{1}{2 + \cos \theta} d\theta &= \int \frac{\frac{2}{1+t^2}}{2 + \frac{1-t^2}{1+t^2}} dt \\ &= \int \frac{2}{2 + 2t^2 + 1 - t^2} dt \\ &= 2 \int \frac{1}{t^2 + 3} dt \\ &= \frac{2}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + \text{常數} \\ &= \frac{2}{\sqrt{3}} \tan^{-1} \frac{\tan \frac{\theta}{2}}{\sqrt{3}} + \text{常數}\end{aligned}$$

1M

1M

1A

7. (a) $A^2 = \begin{pmatrix} 4 & -7 \\ 3 & -5 \end{pmatrix}$ 及 $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

因此， n 的最小值為 3。

1M+1A

1A

- (b) (i) $A + A^2 + A^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

1A

- (ii) $\sum_{k=1}^{335} A^k = (A + A^2 + A^3) + A^3(A + A^2 + A^3) + \dots + A^{333}(A + A^2)$

1M

$$= A + A^2$$

$$= -A^3 = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

1M+1A

解		分
8. (a)	(i) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ h & 2 & -2 \end{vmatrix} = 4 + 2h + 2 + 2h + 2 - 4$	
	$= 4h + 4$	1A
	(E) 有唯一的解當 $4h + 4 \neq 0$ ，即 $h \neq -1$ 。	1A
	因此， $h < -1$ 或 $h > -1$ 。	
	(ii) $x = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 2 \\ k & 2 & -2 \end{vmatrix}}{4h + 4}$	1M
	$= \frac{4k + 12}{4h + 4}$	
	$= \frac{k + 3}{h + 1}$	1A
	(b) 代 $h = -1$ ，(E) 的增廣矩陣為	
	$\left(\begin{array}{ccc c} 1 & 1 & 1 & 2 \\ 1 & -2 & 2 & 3 \\ -1 & 2 & -2 & k \end{array} \right) \sim \left(\begin{array}{ccc c} 1 & 1 & 1 & 2 \\ 0 & -3 & 1 & 1 \\ 0 & 3 & -1 & k+2 \end{array} \right) \sim \left(\begin{array}{ccc c} 1 & 1 & 1 & 2 \\ 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & k+3 \end{array} \right)$	1M
	若 (E) 有解，則 $k = -3$ 。	1A
	設 $z = t$ ，則 $y = \frac{t-1}{3}$ 及 $x = \frac{7-4t}{3}$ 。	
	解集為 $\left\{ \left(\frac{7-4t}{3}, \frac{t-1}{3}, t \right) : t \in \mathbb{R} \right\}$ 。	1A

解	分										
9. (a) 垂直漸近線為 $x = -1$ 及 $x = 3$ 。 水平漸近線為 $y = 2$ 。	1A+1A 1A										
(b) $f'(x) = -\frac{1}{(x+1)^2} + \frac{1}{(x-3)^2}$ 當 $f'(x) = 0$ ， $-\frac{1}{(x+1)^2} + \frac{1}{(x-3)^2} = 0$ $(x-3)^2 = (x+1)^2$ $-8x + 8 = 0$ $x = 1$ <table><tr><td>x</td><td>$x < -1$</td><td>$-1 < x < 1$</td><td>$1 < x < 3$</td><td>$x > 3$</td></tr><tr><td>$f'(x)$</td><td>-</td><td>-</td><td>+</td><td>+</td></tr></table> 極小點為 $(1, f(1))$ ，即 $(1, 3)$ 。	x	$x < -1$	$-1 < x < 1$	$1 < x < 3$	$x > 3$	$f'(x)$	-	-	+	+	1A 1M 1M
x	$x < -1$	$-1 < x < 1$	$1 < x < 3$	$x > 3$							
$f'(x)$	-	-	+	+							
(c) (i) $f''(x) = \frac{2}{(x+1)^3} - \frac{2}{(x-3)^3}$ (ii) 當 $f''(x) = 0$ ， $\frac{2}{(x+1)^3} = \frac{2}{(x-3)^3}$ $(x-3)^3 = (x+1)^3$ $x-3 = x+1$ 無解。 同意該宣稱。	1A 1M 1A										
(d) $S = \int_4^k \left[2 - \left(2 + \frac{1}{x+1} - \frac{1}{x-3} \right) \right] dx$ $= \left[-\ln x+1 + \ln x-3 \right]_4^k$ $= \ln \frac{k-3}{k+1} + \ln 5$ 由於 $k > 4$ ， $0 < \frac{k-3}{k+1} < 1$ ，故此 $\ln \frac{k-3}{k+1} < 0$ 。 因此， $S = \ln \frac{k-3}{k+1} + \ln 5 < \ln 5$ 。	1M 1M 1 1										

10. (a) $\det B = \cos^2 \theta + \sin^2 \theta = 1$

$$B^{-1} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$C = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$= \begin{pmatrix} 4 \cos \theta + \sin \theta & \cos \theta + 4 \sin \theta \\ -4 \sin \theta + \cos \theta & -\sin \theta + 4 \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$= \begin{pmatrix} 4 \cos^2 \theta + 2 \sin \theta \cos \theta + 4 \sin^2 \theta & -\sin^2 \theta + \cos^2 \theta \\ \cos^2 \theta - \sin^2 \theta & 4 \sin^2 \theta - 2 \sin \theta \cos \theta + 4 \cos^2 \theta \end{pmatrix}$$

$$= \begin{pmatrix} 4 + \sin 2\theta & \cos 2\theta \\ \cos 2\theta & 4 - \sin 2\theta \end{pmatrix}$$

(b) (i) 若 C 為對角矩陣，

$$\cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2}$$

故此， $C = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ 。

(ii) $C^n = \lambda C + \mu I$

$$\begin{pmatrix} 5^n & 0 \\ 0 & 3^n \end{pmatrix} = \begin{pmatrix} 5\lambda + \mu & 0 \\ 0 & 3\lambda + \mu \end{pmatrix}$$

故此， $\begin{cases} 5\lambda + \mu = 5^n \\ 3\lambda + \mu = 3^n \end{cases}$ 。

求解後，可得 $\lambda = \frac{5^n - 3^n}{2}$ 及 $\mu = \frac{5(3^n) - 3(5^n)}{2}$ 。

(iii) $(B^{-1}AB)^n = \lambda B^{-1}AB + \mu I$

$$B^{-1}A^nB = \lambda B^{-1}AB + \mu I$$

$$A^n = B(\lambda B^{-1}AB + \mu I)B^{-1}$$

$$= \lambda A + \mu I$$

$$= \frac{1}{2} \begin{pmatrix} 5^n + 3^n & 5^n - 3^n \\ 5^n - 3^n & 5^n + 3^n \end{pmatrix}$$

1M+1A

1M

1

1A

1A

1M

1M

1A

1M

1A

1A

11. (a) 設 $u = a + b - x$ 。則 $du = -dx$ 。

$$\begin{aligned}\int_a^b f(a + b - x) dx &= - \int_b^a f(u) du \\ &= \int_a^b f(u) du \\ &= \int_a^b f(x) dx\end{aligned}$$

1M

1M

1

$$\begin{aligned}\text{(b)} \quad \int_0^{\frac{\pi}{4}} \ln \sin 2x dx &= \int_0^{\frac{\pi}{4}} \ln \sin \left[2 \left(\frac{\pi}{4} - x \right) \right] dx \\ &= \int_0^{\frac{\pi}{4}} \ln \cos 2x dx\end{aligned}$$

1M

1

(c) 設 $2x = t$ 。則 $2 dx = dt$ 。

1M

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \ln \sin 4x dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln \sin 2t dt \\ &= \frac{1}{2} \left[\int_0^{\frac{\pi}{4}} \ln \sin 2t dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \sin 2t dt \right] \\ &= \frac{1}{2} \left[\int_0^{\frac{\pi}{4}} \ln \sin 2t dt + \int_0^{\frac{\pi}{4}} \ln \sin \left[2 \left(u + \frac{\pi}{4} \right) \right] du \right] \quad \text{where } u = t - \frac{\pi}{4} \\ &= \frac{1}{2} \left[\int_0^{\frac{\pi}{4}} \ln \sin 2t dt + \int_0^{\frac{\pi}{4}} \ln \cos 2u du \right] \\ &= \frac{1}{2} \left[\int_0^{\frac{\pi}{4}} \ln \sin 2x dx + \int_0^{\frac{\pi}{4}} \ln \sin 2x dx \right] \\ &= \int_0^{\frac{\pi}{4}} \ln \sin 2x dx\end{aligned}$$

1M

1M

1

$$\begin{aligned}\text{(d)} \quad \int_0^{\frac{\pi}{4}} \ln \sin 4x dx &= \int_0^{\frac{\pi}{4}} (\ln 2 + \ln \sin 2x + \ln \cos 2x) dx \\ &= \ln 2 \left[x \right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \ln \sin 2x dx + \int_0^{\frac{\pi}{4}} \ln \cos 2x dx \\ &= \frac{\pi \ln 2}{4} + 2 \int_0^{\frac{\pi}{4}} \ln \sin 4x dx\end{aligned}$$

1M

1M

$$\int_0^{\frac{\pi}{4}} \ln \sin 4x dx = -\frac{\pi \ln 2}{4}$$

1A

解	分
12. (a) (i) $\overrightarrow{AB} = -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ 及 $\overrightarrow{AC} = -7\mathbf{i} + 3\mathbf{j} + \mathbf{k}$。 $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 5 & -3 \\ -7 & 3 & 1 \end{vmatrix}$ $= 14\mathbf{i} + 22\mathbf{j} + 32\mathbf{k}$ (ii) $\overrightarrow{DA} = \mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$ $\overrightarrow{DE} = \frac{\overrightarrow{DA} \cdot (14\mathbf{i} + 22\mathbf{j} + 32\mathbf{k})}{ 14\mathbf{i} + 22\mathbf{j} + 32\mathbf{k} ^2} (14\mathbf{i} + 22\mathbf{j} + 32\mathbf{k})$ $= -\frac{7}{71}\mathbf{i} - \frac{11}{71}\mathbf{j} - \frac{16}{71}\mathbf{k}$ (b) (i) $\overrightarrow{BC} = -6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ 設 $\overrightarrow{BF} = t\overrightarrow{BC}$。 $\overrightarrow{DF} = (1-t)\overrightarrow{DB} + t\overrightarrow{DC}$ $= -6t\mathbf{i} + (-2-2t)\mathbf{j} + (1+4t)\mathbf{k}$ $0 = \overrightarrow{DF} \cdot \overrightarrow{BC}$ $= (-6t)(-6) + (-2-2t)(-2) + (1+4t)(4)$ $t = -\frac{1}{7}$ 因此，$\overrightarrow{DF} = \frac{6}{7}\mathbf{i} - \frac{12}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$。 (ii) $\overrightarrow{EF} = \overrightarrow{DF} - \overrightarrow{DE}$ $= \frac{475}{497}\mathbf{i} - \frac{775}{497}\mathbf{j} + \frac{325}{497}\mathbf{k}$ $\overrightarrow{BC} \cdot \overrightarrow{EF} = -6\left(\frac{475}{497}\right) - 2\left(-\frac{775}{497}\right) + 4\left(\frac{325}{497}\right)$ $= 0$ 因此，$\overrightarrow{BC}$ 垂直於 $\overrightarrow{EF}$。 (c) 所求之角為 $\angle DFE$。 $\overrightarrow{DF} \cdot \overrightarrow{EF} = (DF)(EF) \cos \angle DFE$ $\frac{1875}{497} = \frac{3\sqrt{21}}{7} \cdot \frac{25\sqrt{1491}}{497} \cos \angle DFE$ $\cos \angle DFE = \frac{25}{3\sqrt{71}}$ $\angle DFE = \cos^{-1} \frac{25}{3\sqrt{71}}$	1M 1A 1M 1A 1M 1M 1A 1M 1A 1M 1M 1A