M2INT-TRIG-2223-ASM-SET 1-MATH
建議題解

1. $$\frac{\sin\theta}{\csc\theta-\cot\theta}-\frac{\tan\theta}{\csc\theta+\cot\theta}=\frac{\sin\theta(\csc\theta+\cot\theta)-\tan\theta(\csc\theta-\cot\theta)}{\csc^2\theta-\cot^2\theta}\qquad\text{1M}$$
$$=\frac{1+\cos\theta-\frac{1}{\cos\theta}+1}{1}\qquad\text{2M}$$
$$=2+\cos\theta-\frac{1}{\cos\theta}$$
$$=2-\frac{1-\cos^2\theta}{\cos\theta}$$
$$=2-\sec\theta\sin^2\theta\qquad\text{1}$$

2. $$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}+\frac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}$$
$$=\frac{(1+\sin\theta-\cos\theta)^2+(1+\sin\theta+\cos\theta)^2}{[(1+\sin\theta)+\cos\theta][(1+\sin\theta)-\cos\theta]}\qquad\text{1M}$$
$$=\frac{(1+\sin\theta)^2-2(1+\sin\theta)\cos\theta+\cos^2\theta+(1+\sin\theta)^2+2(1+\sin\theta)\cos\theta+\cos^2\theta}{(1+\sin\theta)^2-\cos^2\theta}\qquad\text{1M}$$
$$=\frac{2(1+\sin\theta)^2+2\cos^2\theta}{1+2\sin\theta+\sin^2\theta-\cos^2\theta}$$
$$=\frac{2+4\sin\theta+2\sin^2\theta+2\cos^2\theta}{2\sin\theta+2\sin^2\theta}\qquad\text{1M}$$
$$=\frac{4(1+\sin\theta)}{2\sin\theta(1+\sin\theta)}\qquad\text{1M}$$
$$=\frac{2}{\sin\theta}$$
$$=2\csc\theta\qquad\text{1}$$

3. (a)
$$\sin(x+\alpha)=3\cos(x-\alpha)$$
$$\sin x\cos\alpha+\cos x\sin\alpha=3\cos x\cos\alpha+3\sin x\sin\alpha\qquad\text{1M}$$
$$\frac{\sin x}{\cos x}+\frac{\sin\alpha}{\cos\alpha}=3+\frac{3\sin x\sin\alpha}{\cos x\cos\alpha}$$
$$\tan x+\tan\alpha=3+3\tan x\tan\alpha\qquad\text{1M}$$
$$\tan x(1-3\tan\alpha)=3-\tan\alpha$$
$$\tan x=\frac{3\tan\alpha}{1-3\tan\alpha}\qquad\text{1}$$

(b) $$\sin\left(x+\frac{\pi}{4}\right)=3\cos\left(x-\frac{\pi}{4}\right)$$
$$\tan x=\frac{3-\tan\frac{\pi}{4}}{1-3\tan\frac{\pi}{4}}\qquad\text{1M}$$
$$=\frac{3-1}{1-3(1)}$$
$$=-1$$
$$x=\frac{3\pi}{4}\quad\text{或}\quad\frac{7\pi}{4}\qquad\text{1A}$$

4. (a) $\cot(\angle A + \angle B) = \dfrac{1}{\tan(\angle A + \angle B)}$

$= \dfrac{1 - \tan\angle A \tan\angle B}{\tan\angle A + \tan\angle B}$ 1M

$= \dfrac{1 - (\sqrt{a} + \sqrt{a-1})(\sqrt{a} - \sqrt{a-1})}{(\sqrt{a} + \sqrt{a-1}) + (\sqrt{a} - \sqrt{a-1})}$

$= \dfrac{1 - [a - (a-1)]}{2\sqrt{a}}$

$= 0$ 1A

(b) $\cot(\angle A + \angle B) = 0$

$\angle A + \angle B = 90°$ 1A

$\angle C = 180° - (\angle A + \angle B)$

$= 90°$

因此，$\triangle ABC$ 為直角三角形。 1A

5. (a) $k\cos(\alpha - \beta) = \cos(\alpha + \beta)$

$k(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ 1M

$(k+1)\sin\alpha\sin\beta = (1-k)\cos\alpha\cos\beta$

$(k+1)\tan\alpha\tan\beta = 1-k$

$\tan\alpha\tan\beta = \dfrac{1-k}{1+k}$ 1

(b) $\dfrac{1-k}{1+k} = \tan(2\beta)\tan\beta$

$= \dfrac{2\tan\beta}{1-\tan^2\beta}\cdot\tan\beta$

$= \dfrac{2\tan^2\beta}{1-\tan^2\beta}$ 1A

$(1-k)(1-\tan^2\beta) = 2(1+k)\tan^2\beta$

$(3+k)\tan^2\beta = 1-k$

$\tan^2\beta = \dfrac{1-k}{3+k}$

$\tan\beta = \pm\sqrt{\dfrac{1-k}{3+k}}$ 1A

留意當 $\pi < \alpha < 2\pi$，可得 $\dfrac{\pi}{2} < \beta < \pi$ 及 $\tan\beta < 0$。 1A

因此，$\tan\beta = -\sqrt{\dfrac{1-k}{3+k}}$。 1A

6. (a) $\dfrac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$ 1M

$= \dfrac{\frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta} + \frac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta}}$

$= \dfrac{\cot\alpha - \tan\beta}{\cot\alpha + \tan\beta}$ 1

(b) 可得 $\cos(\alpha+\beta)(\cot\alpha+\tan\beta) = \cos(\alpha-\beta)(\cot\alpha-\tan\beta)$。

$\dfrac{1}{\cos 1^\circ}\sum_{i=1}^{90}\cos(2i-1)^\circ[\cot i^\circ + \tan(i-1)^\circ]$

$= \dfrac{1}{\cos 1^\circ}\sum_{i=1}^{90}\cos[i+(i-1)]^\circ[\cot i^\circ + \tan(i-1)^\circ]$

$= \dfrac{1}{\cos 1^\circ}\sum_{i=1}^{90}\cos[i-(i-1)]^\circ[\cot i^\circ - \tan(i-1)^\circ]$ 1M

$= (\cot 1^\circ + \cot 2^\circ + \cot 3^\circ + \dots + \cot 90^\circ) - (\tan 0^\circ + \tan 1^\circ + \tan 2^\circ + \dots + \tan 89^\circ)$

$= (\tan 89^\circ + \tan 88^\circ + \tan 87^\circ + \dots + \tan 0^\circ) - (\tan 0^\circ + \tan 1^\circ + \tan 2^\circ + \dots + \tan 89^\circ)$ 1M

$= 0$ 1A

7. (a) $\tan(\alpha+\beta)(\tan\alpha+\tan\beta)+2$

$= \dfrac{(\tan\alpha+\tan\beta)^2}{1-\tan\alpha\tan\beta}+2$ 1M

$= \dfrac{\tan^2\alpha + 2\tan\alpha\tan\beta + \tan^2\beta + 2(1-\tan\alpha\tan\beta)}{1-\tan\alpha\tan\beta}$

$= \dfrac{\tan^2\alpha + \tan^2\beta + 2}{1-\tan\alpha\tan\beta}$ 1M

$= \dfrac{(\tan^2\alpha+1)+(\tan^2\beta+1)}{1-\tan\alpha\tan\beta}$

$= \dfrac{\sec^2\alpha + \sec^2\beta}{1-\tan\alpha\tan\beta}$ 1

(b) $\tan\left(\alpha+\frac{\pi}{4}\right)\left(\tan\alpha+\tan\frac{\pi}{4}\right) = -8$

$\tan\left(\alpha+\frac{\pi}{4}\right)\left(\tan\alpha+\tan\frac{\pi}{4}\right)+2 = -6$

$\dfrac{\sec^2\alpha + \sec^2\frac{\pi}{4}}{1-\tan\alpha\tan\frac{\pi}{4}} = -6$ 1M

$\dfrac{1+\tan^2\alpha+2}{1-\tan\alpha} = -6$

$\tan^2\alpha + 3 = -6 + 6\tan\alpha$

$\tan^2\alpha - 6\tan\alpha + 9 = 0$ 1A

$\tan\alpha = 3$

$\alpha \approx 1.25$ 或 4.39 1A

8. (a) $\cos 4\theta = \cos 2(2\theta)$

$= 2\cos^2 2\theta - 1$ 1M

$= 2(2\cos^2\theta - 1)^2 - 1$ 1M

$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$

$= 8\cos^4\theta - 8\cos^2\theta + 1$ 1

(b) 代 $x = \cos 15°$，

$16\cos^4 15° - 16\cos^2 15° + 1 = 2(8\cos^4 15° - 8\cos^2 15° + 1) - 1$

$= 2\cos 4(15°) - 1$ 1M

$= 2\cos 60° - 1$

$= 1 - 1$ 1M

$= 0$

因此，$\cos 15°$ 為 $16x^4 - 16x^2 + 1 = 0$ 的根。 1

(c) $16\cos^4 15° - 16\cos^2 15° + 1 = 0$

$\cos^2 15° = \dfrac{-(-16) \pm \sqrt{(-16)^2 - 4(16)(1)}}{2(16)}$ 1M

$= \dfrac{16 \pm 8\sqrt{3}}{32}$

$= \dfrac{2+\sqrt{3}}{4}$ 或 $\dfrac{2-\sqrt{3}}{4}$ 1A

由於 $\cos 15° > \cos 60° > 0$，可得 $\cos^2 15° > \cos^2 60° = \dfrac{1}{4}$ 及 $\cos^2 15° = \dfrac{2+\sqrt{3}}{4}$。 1A

因此，$\cos 15° = \dfrac{\sqrt{2+\sqrt{3}}}{2}$。 1A

9. (a) $\dfrac{\sin\alpha - \sin\beta}{\cos\alpha - \cos\beta} = \dfrac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$ 1M

$= -\cot\dfrac{\alpha+\beta}{2}$ 1

(b) $3\sin\alpha - 4\cos\beta = 3\sin\beta - 4\cos\alpha$

$\dfrac{3(\sin\alpha - \sin\beta)}{4(\cos\alpha - \cos\beta)} = -1$

$-\dfrac{3}{4}\cot\dfrac{\alpha+\beta}{2} = -1$ 1M

$\cot\dfrac{\alpha+\beta}{2} = \dfrac{4}{3}$ 1A

由於 α 及 β 為銳角，$\dfrac{\alpha+\beta}{2}$ 同為銳角。

$\sin\dfrac{\alpha+\beta}{2} = \dfrac{3}{\sqrt{3^2+4^2}}$ 1M

$= \dfrac{3}{5}$ 1A

10. (a)

$$\tan\left(2\cdot\frac{\pi}{8}\right)=\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$ 1M

$$1=\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

$$1-\tan^2\frac{\pi}{8}=2\tan\frac{\pi}{8}$$

$$\tan^2\frac{\pi}{8}+2\tan\frac{\pi}{8}-1=0$$

因此，$\tan\frac{\pi}{8}$ 為方程 $x^2+2x-1=0$。 1

(b) $x^2+2x-1=0$

$$x=\frac{-2\pm\sqrt{2^2-4(1)(-1)}}{2(1)}$$ 1M

$$=-1\pm\sqrt{2}$$

留意 $\tan\frac{\pi}{8}>0>-1-\sqrt{2}$。 1M

因此，$\tan\frac{\pi}{8}=\sqrt{2}-1$。 1A

11. (a) 當 $n=1$，

左式 $=\cos x$

右式 $=\frac{\sin 2x}{2\sin x}=\frac{2\sin x\cos x}{2\sin x}=\cos x=$ 左式

對 $n=1$，命題為真。 1

假設 $\cos x\cos 2x\cos 4x\ldots\cos\left(2^{k-1}x\right)=\frac{\sin\left(2^k x\right)}{2^k\sin x}$，其中 $\sin x\neq 0$ 及 $k\in\mathbb{Z}^+$。 1M

$$\cos x\cos 2x\cos 4x\ldots\cos\left(2^{k-1}x\right)\cos\left(2^k x\right)$$

$$=\frac{\sin\left(2^k x\right)}{2^k\sin x}\cos\left(2^k x\right)$$ 1M

$$=\frac{1}{2^k\sin x}\left(\frac{2\sin\left(2^k x\right)\cos\left(2^k x\right)}{2}\right)$$

$$=\frac{\sin\left(2^{k+1}x\right)}{2^{k+1}\sin x}$$ 1

對 $n=k+1$，命題為真。

藉數學歸納法，$\forall n\in\mathbb{Z}^+$，命題為真。 1

(b)

$$\cos\frac{\pi}{3}\cos\frac{2\pi}{3}\cos\frac{4\pi}{3}\ldots\cos\frac{256\pi}{3}$$

$$=\cos\frac{\pi}{3}\cos\frac{2\pi}{3}\cos\frac{4\pi}{3}\ldots\cos\frac{2^{9-1}\pi}{3}$$

$$=\frac{\sin\frac{2^9\pi}{3}}{2^9\sin\frac{\pi}{3}}$$ 1M

$$=\frac{\sin\left(170\pi+\frac{2\pi}{3}\right)}{512\sin\frac{\pi}{3}}$$

$$=\frac{\sin\frac{2\pi}{3}}{512\sin\frac{\pi}{3}}$$

$$=\frac{1}{512}$$ 1A

12. (a) $\sin\frac{n\pi}{24}\sin\frac{\pi}{24} = -\frac{1}{2}\left[\cos\left(\frac{n\pi}{24}+\frac{\pi}{24}\right)-\cos\left(\frac{n\pi}{24}-\frac{\pi}{24}\right)\right]$ 1M

$= -\frac{1}{2}\left[\cos\frac{(n+1)\pi}{24}-\cos\frac{(n-1)\pi}{24}\right]$ 1

(b) $\sum_{k=1}^{5}\sin\frac{2k\pi}{24}\sin\frac{\pi}{24} = \sum_{k=1}^{5}\left[-\frac{1}{2}\left[\cos\frac{(2k+1)\pi}{24}-\cos\frac{(2k-1)\pi}{24}\right]\right]$ 1M

$$= -\frac{1}{2}\left[\sum_{k=1}^{5}\cos\frac{(2k+1)\pi}{24}-\sum_{k=1}^{5}\cos\frac{(2k-1)\pi}{24}\right]$$

$$= -\frac{1}{2}\left(\cos\frac{11\pi}{24}-\cos\frac{\pi}{24}\right)$$

$$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{24}\right)-\cos\frac{\pi}{24}\right]$$

$= -\frac{1}{2}\left(\sin\frac{\pi}{24}-\cos\frac{\pi}{24}\right)$ 1A

$\sum_{k=1}^{5}\sin\frac{k\pi}{12} = \frac{\sum_{k=1}^{5}\sin\frac{2k\pi}{24}\sin\frac{\pi}{24}}{\sin\frac{\pi}{24}}$ 1M

$$= \frac{-\frac{1}{2}\left(\sin\frac{\pi}{24}-\cos\frac{\pi}{24}\right)}{\sin\frac{\pi}{24}}$$

$$= -\frac{1}{2}+\frac{1}{2}\cot\frac{\pi}{24}$$

$$= -\frac{1}{2}+\frac{1}{2}(2+\sqrt{2}+\sqrt{3}+\sqrt{6})$$

$= \frac{1+\sqrt{2}+\sqrt{3}+\sqrt{6}}{2}$ 1A

– 試卷完 –