

M2INT-INT-2223-ASM-SET 1-MATH

建議題解

$$\begin{aligned}
 1. \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos 2x}{\cos x + \sin x} dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x} dx \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x) dx && 1M \\
 &= \left[\sin x + \cos x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} && 1A \\
 &= (1 + 0) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) \\
 &= 1 - \sqrt{2} && 1A
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^{\frac{\pi}{4}} \sin^4 x dx &= \frac{1}{4} \int_0^{\frac{\pi}{4}} (1 - \cos 2x)^2 dx && 1M \\
 &= \frac{1}{4} \int_0^{\frac{\pi}{4}} (1 - 2 \cos 2x + \cos^2 2x) dx \\
 &= \frac{1}{4} \int_0^{\frac{\pi}{4}} \left[1 - 2 \cos 2x + \frac{1}{2}(1 + \cos 4x) \right] dx && 1M \\
 &= \frac{1}{4} \left[\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8} \right]_0^{\frac{\pi}{4}} && 1A \\
 &= \frac{1}{4} \left(\frac{3\pi}{8} - 1 + 0 \right) - \frac{1}{4}(0) \\
 &= \frac{3\pi}{32} - \frac{1}{4} && 1A
 \end{aligned}$$

3. (a) 設 $u = 1 - e^{-x}$ 。則 $du = e^{-x} dx$ 。 1M

當 $x = \ln 2$ ， $u = \frac{1}{2}$ ；當 $x = \ln 3$ ， $u = \frac{2}{3}$ 。

$$\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}} dx = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{du}{u} \quad 1M$$

$$= \left[\ln |u| \right]_{\frac{1}{2}}^{\frac{2}{3}} \quad 1A$$

$$= \ln \frac{2}{3} - \ln \frac{1}{2}$$

$$= \ln \frac{4}{3} \quad 1A$$

$$\begin{aligned} \text{(b)} \quad \frac{A}{u} + \frac{B}{u-1} &= \frac{A(u-1) + Bu}{u(u-1)} \\ &= \frac{(A+B)u - A}{u(u-1)} \end{aligned}$$

故此， $A + B = 0$ 及 $-A = 1$ 。 1M

求解後，可得 $A = -1$ 及 $B = 1$ 。 1A

$$\text{(c)} \quad \int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x - 1)} dx = \int_{\ln 2}^{\ln 3} \left(\frac{1}{e^x - 1} - \frac{1}{e^x} \right) dx \quad 1M$$

$$= \int_{\ln 2}^{\ln 3} \left(\frac{e^{-x}}{1 - e^{-x}} - e^{-x} \right) dx$$

$$= \ln \frac{4}{3} + \left[e^{-x} \right]_{\ln 2}^{\ln 3} \quad 1M+1A$$

$$= \ln \frac{4}{3} + \left(\frac{1}{3} - \frac{1}{2} \right)$$

$$= \ln \frac{4}{3} - \frac{1}{6} \quad 1A$$

4. 設 $u = \frac{3}{2} - \frac{2}{x}$ 。則 $du = \frac{2}{x^2} dx$ 。 1M

$$\int \frac{dx}{x^2 \sqrt{\frac{3}{2} - \frac{2}{x}}} = \frac{1}{2} \int \frac{du}{\sqrt{u}} \quad 1M$$

$$= u^{\frac{1}{2}} + C$$

$$= \left(\frac{3}{2} - \frac{2}{x} \right)^{\frac{1}{2}} + C \quad 1A$$

5. 設 $u = \frac{1}{x}$ 。則 $du = -\frac{1}{x^2} dx$

$$\int \frac{\sqrt[3]{4x^3 - 2}}{x^5} dx = \int \sqrt[3]{4 - \frac{2}{x^3}} \cdot \frac{1}{x^4} dx \quad 1M$$

$$= - \int \sqrt[3]{4 - 2u^3} u^2 du \quad 1M$$

設 $w = 4 - 2u^3$ 。則 $dw = -6u^2 du$ 。 1M

$$\int \frac{\sqrt[3]{4x^3 - 2}}{x^5} dx = \frac{1}{6} \int \sqrt[3]{w} dw \quad 1M$$

$$= \frac{1}{8} w^{\frac{4}{3}} + \text{常數}$$

$$= \frac{1}{8} \left(4 - \frac{2}{x^3} \right)^{\frac{4}{3}} + \text{常數} \quad 1A$$

6. 設 $u = 1 - 2x^2$ 。則 $du = -4x dx$ 。 1M

$$\int \frac{2x^3 dx}{(1 - 2x^2)^{\frac{1}{3}}} = -\frac{1}{4} \int \frac{1 - u}{u^{\frac{1}{3}}} du \quad 1M$$

$$= -\frac{1}{4} \int (u^{-\frac{1}{3}} - u^{\frac{2}{3}}) du \quad 1M$$

$$= -\frac{3}{8} u^{\frac{2}{3}} + \frac{3}{20} u^{\frac{5}{3}} + \text{常數}$$

$$= -\frac{3}{8} (1 - 2x^2)^{\frac{2}{3}} + \frac{3}{20} (1 - 2x^2)^{\frac{5}{3}} + \text{常數} \quad 1A$$

7. (a) 設 $u = x + 2$ 。則 $du = dx$ 。 1M

$$\int \frac{x - 1}{x + 2} dx = \int \frac{u - 3}{u} du$$

$$= \int \left(1 - \frac{3}{u} \right) du$$

$$= u - 3 \ln |u| + \text{常數}$$

$$= x - 3 \ln |x + 2| + \text{常數} \quad 1A$$

(b) 設 $u = \sqrt{x}$ 。則 $du = \frac{1}{2\sqrt{x}} dx$ 。 1M

$$\int \frac{\sqrt{x} - 1}{x + 2\sqrt{x}} dx = 2 \int \left(\frac{\sqrt{x} - 1}{\sqrt{x} + 2} \cdot \frac{1}{2\sqrt{x}} \right) dx$$

$$= 2 \int \frac{u - 1}{u + 2} du \quad 1A$$

$$= 2u - 6 \ln |u + 2| + \text{常數} \quad 1A$$

$$= 2\sqrt{x} - 6 \ln(\sqrt{x} + 2) + \text{常數} \quad 1A$$

8. (a) 設 $x = 3 \tan \theta$ 。則 $dx = 3 \sec^2 \theta d\theta$ 。 1M

當 $x = 0$ ， $\theta = 0$ ；當 $x = 1$ ， $\theta = \tan^{-1} \frac{1}{3}$ 。

$$\int_0^1 \frac{dx}{x^2 + 9} = \int_0^{\tan^{-1} \frac{1}{3}} \frac{3 \sec^2 \theta d\theta}{9 \tan^2 \theta + 9} \quad 1M$$

$$= \frac{1}{3} \int_0^{\tan^{-1} \frac{1}{3}} d\theta$$

$$= \frac{1}{3} \left[\theta \right]_0^{\tan^{-1} \frac{1}{3}} \quad 1A$$

$$= \frac{1}{3} \tan^{-1} \frac{1}{3} \quad 1A$$

(b) $\int_0^1 \frac{x^2}{x^2 + 9} dx = \int_0^1 \left(1 - \frac{9}{x^2 + 9} \right) dx \quad 1M$

$$= \left[x \right]_0^1 - 9 \times \frac{1}{3} \tan^{-1} \frac{1}{3}$$

$$= 1 - 3 \tan^{-1} \frac{1}{3} \quad 1A$$

9. 設 $x = \sqrt{3} \tan \theta$ 。則 $dx = \sqrt{3} \sec^2 \theta d\theta$ 。 1M

$$\int_{-3}^3 \frac{1}{(x^2 + 3)^{\frac{3}{2}}} dx = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3} \sec^2 \theta}{(3 \tan^2 \theta + 3)^{\frac{3}{2}}} d\theta \quad 1A$$

$$= \frac{1}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2 \theta}{\sec^3 \theta} d\theta$$

$$= \frac{1}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos \theta d\theta$$

$$= \frac{1}{3} \left[\sin \theta \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{3} \quad 1A$$

10. 設 $x = \sin^2 \theta$ ，其中 $0 \leq \theta \leq \frac{\pi}{2}$ 。則 $dx = 2 \sin \theta \cos \theta d\theta$ 。

當 $x = 0$ ， $\theta = 0$ ；當 $x = \frac{3}{4}$ ， $\theta = \frac{\pi}{3}$ 。 1A

$$\int_0^{\frac{3}{4}} \sqrt{\frac{x}{1-x}} dx = \int_0^{\frac{\pi}{3}} \sqrt{\frac{\sin^2 \theta}{1 - \sin^2 \theta}} \cdot 2 \sin \theta \cos \theta d\theta \quad 1M$$

$$= \int_0^{\frac{\pi}{3}} 2 \sin^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{3}} (1 - \cos 2\theta) d\theta \quad 1M$$

$$= \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{3}} \quad 1A$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4} \quad 1A$$

11. (a) $\int_{-1}^1 \frac{dx}{x^2 + 4x + 7} = \int_{-1}^1 \frac{dx}{(x+2)^2 + 3}$ 1M

設 $x+2 = \sqrt{3} \tan \theta$ 。則 $dx = \sqrt{3} \sec^2 \theta d\theta$ 。 1M

當 $x = -1$ ， $\theta = \frac{\pi}{6}$ ；當 $x = 1$ ， $\theta = \frac{\pi}{3}$ 。

$$\int_{-1}^1 \frac{dx}{x^2 + 4x + 7} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{3} \sec^2 \theta}{3 \sec^2 \theta} d\theta$$
 1M

$$= \frac{1}{\sqrt{3}} \left[\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
 1A

$$= \frac{\sqrt{3}\pi}{18}$$
 1A

(b) 設 $u = x^2 + 4x + 7$ 。則 $dx = (2x+4) dx$ 。 1M

當 $x = -1$ ， $u = 4$ ；當 $x = 1$ ， $u = 12$ 。

$$\int_{-1}^1 \frac{x}{x^2 + 4x + 7} dx = \frac{1}{2} \int_{-1}^1 \frac{2x+4}{x^2 + 4x + 7} dx - \int_{-1}^1 \frac{2}{x^2 + 4x + 7} dx$$
 1M

$$= \frac{1}{2} \int_4^{12} \frac{du}{u} - 2 \times \frac{\sqrt{3}\pi}{18}$$
 1M

$$= \frac{1}{2} \left[\ln |u| \right]_4^{12} - \frac{\sqrt{3}\pi}{9}$$
 1A

$$= \frac{1}{2} \ln 3 - \frac{\sqrt{3}\pi}{9}$$
 1A

12. (a) 設 $u = e^{3x} + 3x + 1$ 。則 $du = 3(e^{3x} + 1) dx$ 。 1M

當 $x = 1$ ， $u = e^3 + 4$ ；當 $x = 4$ ， $u = e^{12} + 13$ 。

$$\int_1^4 \frac{e^{3x} + 1}{e^{3x} + 3x + 1} dx = \frac{1}{3} \int_{e^3+4}^{e^{12}+13} \frac{du}{u}$$
 1M

$$= \frac{1}{3} \left[\ln |u| \right]_{e^3+4}^{e^{12}+13}$$
 1A

$$= \frac{1}{3} \ln \frac{e^{12} + 13}{e^3 + 4}$$
 1A

(b) $\int_1^4 \frac{x dx}{e^{3x} + 3x + 1} = \frac{1}{3} \int_1^4 \frac{e^{3x} + 3x + 1 - (e^{3x} + 1)}{e^{3x} + 3x + 1} dx$ 1M

$$= \frac{1}{3} \left[x \right]_1^4 - \frac{1}{3} \times \frac{1}{3} \ln \frac{e^{12} + 13}{e^3 + 4}$$
 1M

$$= 1 - \frac{1}{9} \ln \frac{e^{12} + 13}{e^3 + 4}$$
 1A

13. (a) 設 $x = \sin \theta$ 。則 $dx = \cos \theta d\theta$ 。

當 $x = 0$ ， $\theta = 0$ ；當 $x = 1$ ， $\theta = \frac{\pi}{2}$ 。

$$\int_0^1 \sqrt{1-x^2} dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 \theta} \cos \theta d\theta \quad 1M$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta \quad 1M$$

$$= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} \quad 1A$$

$$= \frac{1}{2} \left(\frac{\pi}{2} + 0 - 0 - 0 \right) \quad 1A$$

$$= \frac{\pi}{4}$$

(b) $\frac{d}{dx}[(1-x^2)^{\frac{3}{2}}x] = (1-x^2)^{\frac{3}{2}} + \frac{3}{2}(1-x^2)^{\frac{1}{2}}(-2x)(x) \quad 1M$

$$= \sqrt{1-x^2}[(1-x^2) - 3x^2]$$

$$= \sqrt{1-x^2} - 4x^2\sqrt{1-x^2} \quad 1$$

(c) 利用 (b)， $4x^2\sqrt{1-x^2} = \sqrt{1-x^2} - \frac{d}{dx}[(1-x^2)^{\frac{3}{2}}x]$ 。

$$\int_0^1 x^2\sqrt{1-x^2} dx = \frac{1}{4} \int_0^1 \sqrt{1-x^2} dx - \frac{1}{4} \left[(1-x^2)^{\frac{3}{2}}x \right]_0^1 \quad 1M$$

$$= \frac{1}{4} \times \frac{\pi}{4} - \frac{1}{4}(0-0)$$

$$= \frac{\pi}{16} \quad 1A$$