M2REG-SPVP-2223-ASM-SET 1-MATH
Suggested solutions

1. $\mathbf{p} \cdot \mathbf{c} = 0$ 1M

$k(\mathbf{a} \cdot \mathbf{c}) + 2(\mathbf{b} \cdot \mathbf{c}) = 0$

$4k - 2 = 0$

$k = \frac{1}{2}$ 1A

2. (a) $\mathbf{a} \cdot \mathbf{b} = (5)(2)\cos 120° = -5$ 1A

(b) $(\mathbf{a} + k\mathbf{b}) \cdot (\mathbf{a} + 3\mathbf{b}) = 0$ 1M

$|\mathbf{a}|^2 + (3+k)\mathbf{a} \cdot \mathbf{b} + 3k|\mathbf{b}|^2 = 0$

$5^2 + (3+k)(-5) + 3k(4) = 0$

$k = -\frac{10}{7}$ 1A

3. (a) $\overrightarrow{OP} = \frac{1}{r+1}\overrightarrow{OB} + \frac{r}{r+1}\overrightarrow{OA}$ 1M

$= \frac{1}{r+1}(5\mathbf{i} + \mathbf{j}) + \frac{r}{r+1}(2\mathbf{i} + 4\mathbf{j})$

$= \frac{5+2r}{r+1}\mathbf{i} + \frac{4r+1}{r+1}\mathbf{j}$ 1A

(b) $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$ 1M

$\left(\frac{5+2r}{r+1}\mathbf{i} + \frac{4r+1}{r+1}\mathbf{j}\right) \cdot (3\mathbf{i} - 3\mathbf{j}) = 0$

$\frac{1}{r+1}[(15+6r) - (12r+3)] = 0$

$r = 2$ 1A

4. $\overrightarrow{AB} = (2v - 2u - 1)\mathbf{i} + (v + 3u - 2)\mathbf{j} + (-v - u - 2)\mathbf{k}$ 1A

$\overrightarrow{AB} \cdot (\mathbf{j} + \mathbf{k}) = 0$ 1M

$v + 3u - 2 + (-v - u - 2) = 0$

$u = 2$ 1A

$\overrightarrow{AB} \cdot (-7\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 0$

$-7(2v - 5) - (v + 4) - 2(-v - 4) = 0$

$v = 3$ 1A

5. $(2\sqrt{3})^2 = |\mathbf{a} - \mathbf{b}|^2$

$$\begin{aligned} 12 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) && \text{1M} \\ &= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \\ &= 5^2 - 2\mathbf{a} \cdot \mathbf{b} + 3^2 \end{aligned}$$

$\mathbf{a} \cdot \mathbf{b} = 11$ 1A

$$\begin{aligned} (2\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - 4\mathbf{b}) &= 2|\mathbf{a}|^2 - 7\mathbf{a} \cdot \mathbf{b} - 4|\mathbf{b}|^2 \\ &= 2(5)^2 - 7(11) - 4(3)^2 \\ &= -63 \neq 0 \end{aligned}$$

Thus, $2\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - 4\mathbf{b}$ are not orthogonal vectors. 1A

6. $|3\mathbf{a} - 4\mathbf{b}|^2 = 5^2$ and $|3\mathbf{a} + 4\mathbf{b}|^2 = 11^2$

$(3\mathbf{a} - 4\mathbf{b}) \cdot (3\mathbf{a} - 4\mathbf{b}) = 25$ $(3\mathbf{a} + 4\mathbf{b}) \cdot (3\mathbf{a} + 4\mathbf{b}) = 121$ 1M

$9|\mathbf{a}|^2 - 24\mathbf{a} \cdot \mathbf{b} + 16|\mathbf{b}|^2 = 25$ $9|\mathbf{a}|^2 + 24\mathbf{a} \cdot \mathbf{b} + 16|\mathbf{b}|^2 = 121$ 1A

Therefore,

$121 - 25 = 48\mathbf{a} \cdot \mathbf{b}$ 1M

$\mathbf{a} \cdot \mathbf{b} = 2$ 1A

7. (a) $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$

$= |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$ 1M

$= 4 + 2k + 4$

$|\mathbf{a} + \mathbf{b}| = \sqrt{8 + 2k}$ 1A

(b) $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a}||\mathbf{a} + \mathbf{b}| \cos 30°$ 1M

$|\mathbf{a}|^2 + \mathbf{a} \cdot \mathbf{b} = 2\sqrt{8 + 2k} \cos 30°$

$4 + k = \sqrt{8 + 2k}\sqrt{3}$

$k^2 + 8k + 16 = 24 + 6k$

$k^2 + 2k - 8 = 0$ 1M

$k = 2$ or -4

When $k = -4$, $\mathbf{a} + \mathbf{b}$ is a zero vector and is not possible to have an angle 30° with $\mathbf{a}$. 1A

$|\mathbf{a} + \mathbf{b}| = \sqrt{8 + 2(2)} = 2\sqrt{3} > 2 = |\mathbf{a}|$

The claim is agreed. 1A

8. (a) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -\mathbf{i} - 7\mathbf{j}$ 1A

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = 7\mathbf{i} - 4\mathbf{j}$ 1A

(b) $\overrightarrow{PQ} \cdot \overrightarrow{PR} = \left|\overrightarrow{PQ}\right|\left|\overrightarrow{PR}\right| \cos \angle QPR$

$-7 + 28 = \sqrt{1^2 + 7^2}\sqrt{7^2 + 4^2} \cos \angle QPR$ 1M

$\angle QPR \approx 68°$ 1A

9. (a) $\overrightarrow{OC} = \overrightarrow{OB} - \overrightarrow{OA} = 4\mathbf{i} + 2\mathbf{j}$ 1A

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 7\mathbf{i} - \mathbf{j}$ 1A

(b) $\overrightarrow{OB} \cdot \overrightarrow{AC} = (OB)(AC)|\cos\theta|$ 1M

$2 = \sqrt{1^2 + 5^2}\sqrt{7^2 + 1^2}|\cos\theta|$ 1A

$\theta \approx 93°$ or $87°$ (rejected) 1A

10. $\overrightarrow{AP} = \frac{1}{4}\overrightarrow{AC} + \frac{3}{4}\overrightarrow{AB}$

$= \frac{19}{4}\mathbf{i} + 5\mathbf{j}$ 1A

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = -9\mathbf{i} + 4\mathbf{j}$ 1A

$\overrightarrow{AP} \cdot \overrightarrow{BC} = (AP)(BC)\cos\angle APQ$

$-\frac{91}{4} = \sqrt{\left(\frac{19}{4}\right)^2 + 5^2}\sqrt{9^2 + 4^2}\cos\angle APQ$ 1M

$\angle APQ \approx 110°$ 1A

11. (a) $\overrightarrow{OP} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ 1A

(b) (i) $\overrightarrow{OP} \cdot \overrightarrow{OP} = 3^2$

$\frac{4}{9}|\mathbf{a}|^2 + \frac{1}{9}|\mathbf{b}|^2 + \frac{4}{9}\mathbf{a} \cdot \mathbf{b} = 9$ 1M

$4 + 4 + \frac{4}{9}\mathbf{a} \cdot \mathbf{b} = 9$

$\mathbf{a} \cdot \mathbf{b} = \frac{9}{4}$ 1A

(ii) $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\angle AOB$ 1M

$\frac{9}{4} = (3)(6)\cos\angle AOB$

$\angle AOB = \cos^{-1}\frac{1}{8}$ 1A

12. (a) $\overrightarrow{OR} = \frac{1}{4}\overrightarrow{OP} + \frac{3}{4}\overrightarrow{OQ}$ 1M

$= 10\mathbf{i} - 2\mathbf{j}$ 1A

$OR = \sqrt{10^2 + 2^2} = \sqrt{104}$ 1A

Required unit vector $= \frac{1}{\sqrt{104}}(10\mathbf{i} - 2\mathbf{j})$ 1M

$= \frac{1}{\sqrt{26}}(5\mathbf{i} - \mathbf{j})$ 1A

(b) $(4\mathbf{i} + \mathbf{j}) \cdot \left(\frac{1}{\sqrt{26}}(5\mathbf{i} - \mathbf{j})\right) = \sqrt{4^2 + 1^2}(1)\cos\angle ROP$ 1M

$\cos\angle ROP = \frac{19}{\sqrt{26} \times \sqrt{17}}$ 1M

$\angle ROP \approx 25°$ 1A

13. (a)

$$0 = \overrightarrow{OP} \cdot \overrightarrow{CM}$$

$$= 4\mathbf{a} \cdot [(2-r)\mathbf{a} - s\mathbf{b}] \qquad \text{1M}$$

$$= 4\left[(2-r)(1) - s\left(\frac{1}{3}\right)\right] \qquad \text{1M}$$

$$3r + s = 6 \qquad \text{1}$$

(b) Let R be the midpoint of OQ. Then $\overrightarrow{OR} = \mathbf{b}$.

$$0 = \overrightarrow{OR} \cdot \overrightarrow{CR}$$

$$= \mathbf{b} \cdot [-r\mathbf{a} + (1-s)\mathbf{b}] \qquad \text{1M}$$

$$= -r\left(\frac{1}{3}\right) + (1-s)(1)$$

$$r + 3s = 3 \qquad \text{1A}$$

Solving, we have $r = \frac{15}{8}$ and $s = -\frac{3}{8}$. 1A

(c) $\overrightarrow{CN} = (2\mathbf{a} + \mathbf{b}) - \left(\frac{15}{8}\mathbf{a} + \frac{3}{8}\mathbf{b}\right)$ 1M

$$= \frac{1}{8}\mathbf{a} + \frac{5}{8}\mathbf{b} \qquad \text{1A}$$

Since $OH \perp PQ$ and $CN \perp PQ$, we have $OH // CN$.

$$\overrightarrow{OH} = k\left(\frac{1}{8}\mathbf{a} + \frac{5}{8}\mathbf{b}\right) \qquad \text{1M}$$

$$\overrightarrow{QH} = \frac{k}{8}\mathbf{a} + \left(\frac{5k}{8} - 2\right)\mathbf{b} \qquad \text{1A}$$

Since $QH \perp OP$,

$$0 = \overrightarrow{QH} \cdot \overrightarrow{OP}$$

$$= \frac{4k}{8}\mathbf{a} \cdot \mathbf{a} + 4\left(\frac{5k}{8} - 2\right)\mathbf{a} \cdot \mathbf{b} \qquad \text{1M}$$

$$= \frac{4k}{8} + \frac{4}{3}\left(\frac{5k}{8} - 2\right)$$

$$= \frac{4(8k - 16)}{8}$$

$$k = 2 \qquad \text{1A}$$

Thus, $\overrightarrow{OH} = 2\left(\frac{1}{8}\mathbf{a} + \frac{5}{8}\mathbf{b}\right) \frac{1}{4}\mathbf{a} + \frac{5}{4}\mathbf{b}$. 1A

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