REG-COT-2223-ASM-SET 1-MATH
Suggested solutions

Multiple Choice Questions

1. B

   I. ✗. Orthocentre is at B.
   II. ✓. Centroid always lies inside the triangle.
   III. ✗. Incentre always lies inside the triangle.

2. C

   No steps required.

3. D

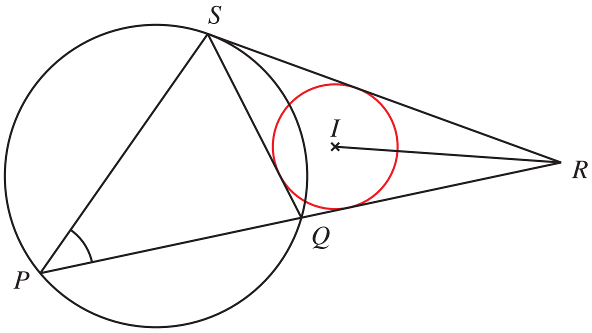
   I. ✓. AD is the perpendicular bisector of BC.
   II. ✓. AD is the altitude through A.
   III. ✓. AD is the median through A.

4. C

   I is the incentre of $\triangle QRS$.

   $\angle IRQ = \angle IRS = 12°$
   $\angle SRQ = 12° + 12° = 24°$
   $\angle QPS = \angle SQR$
   In $\triangle PSR$,

   $$70° + 2\angle QPS + 24° = 180°$$
   $$\angle QPS = 43°$$



5. B

   $BC = 2BL = 26\,\text{cm}$
   $AB = 2BN = 10\,\text{cm}$
   $AC = 2CM = 24\,\text{cm}$
   Since $10^2 + 24^2 = 26^2$, $\angle BAC = 90°$.
   Required area $= \dfrac{(10)(24)}{2} = 120\,\text{cm}^2$

6. $\boxed{\text{C}}$

Let the centre of circle be O.
BO is a median of $\triangle ABC$. So, BEO is a straight line.
$\angle BAC = \angle ABE = 27°$
$\angle CBD = \angle BAC = 27°$
$\angle ABC = 90°$
In $\triangle ABD$,

$$x + 27° + (90° + 27°) = 180°$$
$$x = 36°$$

7. $\boxed{\text{B}}$

Let $AC = 2$. Then $BC = 2$, $CD = 1$ and $AE = \dfrac{AB}{2} = \sqrt{2}$
Note that $CE \perp AB$, $\angle ACE = \sin^{-1}\dfrac{\sqrt{2}}{2} = 45°$
$\angle CAD = \tan^{-1}\dfrac{1}{2} \approx 26.6°$
$\theta = 180° - 45° - \tan^{-1}\dfrac{1}{2} \approx 108°$
$\sin\theta \approx 0.948683$.
By checking all options, we see that $\sin\theta = \dfrac{3\sqrt{10}}{10}$

8. $\boxed{\text{A}}$

I. ✓. Let the radius of inscribed circle of $\triangle ABC$ be r.
Both angles are equal to $\tan^{-1}\dfrac{OV}{r}$.
II. ✓. Let the radius of circumcircle ABC be R.
Then $OB = OC = R$ and both angles are equal to $\tan^{-1}\dfrac{OV}{R}$.
III. ✗. If $\angle ABC = 90°$, then points O and B coincide.
Angle between plane VAB and plane ABC is 90°, while angle between plane VAC and plane ABC is not 90°.

9. $\boxed{\text{D}}$

y-coordinate of circumcentre $= \dfrac{(-2) + (8)}{2} = 3$
Let the coordinates of the circumcentre be $(h,\ 3)$.

$$\sqrt{(h-2)^2 + (3-8)^2} = \sqrt{(h-10)^2 + (14-3)^2}$$
$$h^2 - 4h + 29 = h^2 - 20h + 221$$
$$16h = 192$$
$$h = 12$$

x-coordinate of the circumcentre is 12.

10. A

Point B can be obtained by rotating A through 90° anticlockwise about the origin. So, $\angle AOB = 90°$.
Circumcentre is the midpoint of AB, required y-coordinate $= \dfrac{8-2}{2} = 3$

11. A

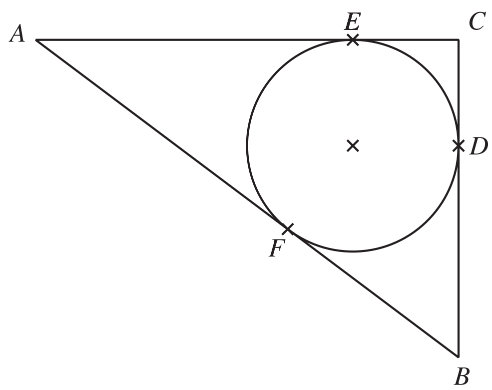
Since AB is parallel to the y-axis, altitude through O is parallel to the x-axis.
Let the coordinates of orthocentre H be $(x,\ 0)$. Since $AH \perp OB$,

$$\frac{12-0}{16-x} \times \frac{-12-0}{16-0} = -1$$
$$x = 7$$

12. C

Let D, E, F be the point of contact of the inscribed circle and BC, AC, AB respectively.

Let $CE = x$. Then $AE = 12 - x$, $CD = x$ and $BD = 5 - x$.
By tangent properties, $AF = 12 - x$ and $BF = 5 - x$.
Since $AB = \sqrt{5^2 + 12^2} = 13$, we have $x = 2$.
Thus, the coordinates of incentre are $(10,\ 3)$.



13. A

Circumcentre lies on perpendicular bisector of OA.
y-coordinate of the circumcentre $= \dfrac{0+12}{2} = 6$

14. A

Altitude through Q is horizontal.
y-coordinate of the orthocentre = 48
Let the coordinates of the orthocentre be $(x,\ 48)$.

$$\frac{48-0}{x-0} \times \frac{60-48}{0-96} = -1$$
$$x = 6$$

15. B

Observe that $\angle BAC = 90°$.

I. ✗. Circumcentre is at the midpoint of BC.

II. ✓. x-coordinate of centroid is 1 by symmetry.

III. ✓. Orthocentre is at A (because it is a right-angled triangle).

16. B

The coordinates of the vertices of the triangle are $(0,\ 0)$, $(6,\ 0)$ and $(0,\ 8)$.

Let the radius of inscribed circle be r.

By considering the area of the triangle,

$$\frac{(6)(8)}{2} = \frac{(6)(r)}{2} + \frac{(8)(r)}{2} + \frac{(\sqrt{6^2+8^2})(r)}{2}$$

$$r = 2$$

The coordinates of the incentre are $(2,\ 2)$.

17. D

Denote the incentre by X.

Let D, E and F be points on OA, OB and AB respectively such that $XD \perp OA$, $XE \perp OB$ and $XF \perp AB$.

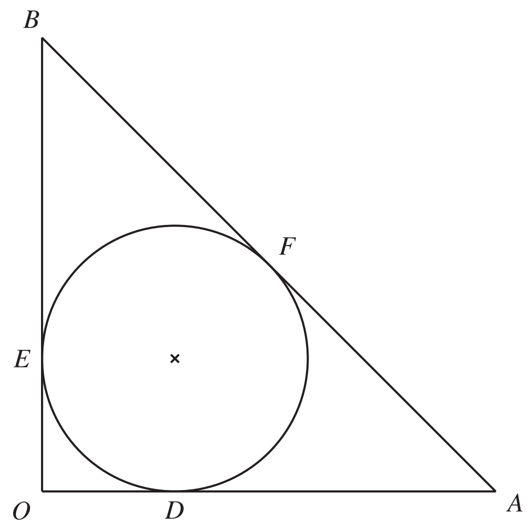
Let the radius be r.

$OD = OE = r$.

$AD = AF = 6 - r$ and $BE = BF = 6 - r$.

$$(6-r) + (6-r) = \sqrt{6^2+6^2}$$

$$r = 6 - 3\sqrt{2}$$



18. D

Altitude through B is vertical.

The x-coordinate of the orthocentre is 24.

Let the coordinates of the orthocentre be $(24,\ y)$.

$$\frac{y-0}{24-0} \times \frac{18-0}{24-48} = -1$$

$$y = 32$$

19. D

Since OA is vertical, orthocentre lies on the horizontal line through B

Thus, y-coordinate of orthocentre = -12.

Let the coordinates of orthocentre H be $(x,\ -12)$. As $AH \perp BO$,

$$\frac{36+12}{0-x} \times \frac{12}{16} = -1$$
$$x = 36$$

20. D

Let $H\ (h,\ k)$ be the orthocentre.

$AH \perp BC$ and $BH \perp AC$

$$\frac{k+19}{h+38} \times \frac{9+1}{-10+2} = -1 \qquad \frac{k-9}{h+10} \times \frac{-1+19}{-2+38} = -1$$
$$\frac{k+19}{h+38} = \frac{4}{5} \qquad \frac{k-9}{h+10} = -2$$

Solving, we have $h = -8$ and $k = 5$.

21. D

x-coordinate of vertex = $\dfrac{-k}{2(1)} = -\dfrac{k}{2}$

Midpoint of $PR = \left(-\dfrac{k}{2},\ 0\right)$

Consider the x-coordinate of centroid,

$$-2 = \frac{0(1) + \left(-\frac{k}{2}\right)(2)}{1+2}$$
$$k = 6$$

22. C

Let the coordinates of Q be $(p,\ q)$. Denote the orthocentre of $\triangle OPQ$ by H.

$OP \perp QH$ and $PH \perp OQ$

$$\frac{-18}{26} \times \frac{q-21}{p+3} = -1 \qquad \frac{-18+3}{26-21} \times \frac{q}{p} = -1$$
$$13p - 9q = 300 \qquad p - 3q = 0$$

Solving, we have $p = 30$ and $q = 10$.

Thus, y-coordinate of Q is 10.

23. B

$\triangle OAB$ is a right-angled triangle. So, orthocentre of $\triangle OAB$ is at point O.

Area of $\triangle OAB = \frac{1}{2}(12)(16) = 96$.

$AB = \sqrt{12^2 + 16^2} = 20$. Let the required distance be d. By considering the area of $\triangle OAB$,

$$\frac{1}{2}(AB)(d) = 96$$
$$d = 9.6$$

24. C

The straight line $x - 2y + 10 = 0$ is perpendicular to the straight line $2x + y + a = 0$.

The triangle formed is a right-angled triangle. The orthocentre lies at the right-angled vertex.

When $x = -6$,

$$(-6) - 2y + 10 = 0$$
$$y = 2$$

Substitute $(-6,\ 2)$ into $2x + y + a = 0$,

$$2(-6) + (2) + a = 0$$
$$a = 10$$

25. A

$P\left(\frac{k}{2},\ 0\right)$, $Q\ (0,\ -k)$ and $R\ (0,\ k)$.

Circumcentre lies on perpendicular bisector of QR, i.e., the x-axis.

Circumcentre is at $(-3,\ 0)$.

$$\frac{k}{2} - (-3) = \sqrt{3^2 + k^2}$$
$$0 = \frac{3k^2}{4} - 3k$$
$$k = 4 \quad \text{or} \quad 0$$

y-coordinate of $R = 4$

26. B

$A = \left(\frac{k}{3},\ 0\right)$ and $B = \left(0,\ \frac{k}{4}\right)$

Circumcentre lies on perpendicular bisector of AC. So $h = \frac{k + \frac{k}{3}}{2} = \frac{2k}{3}$.
Circumcentre is equidistant from B and A,

$$\sqrt{\left(\frac{2k}{3} - k\right)^2 + 38^2} = \sqrt{\left(\frac{2k}{3}\right)^2 + \left(38 - \frac{k}{4}\right)^2}$$

$$-\frac{4k^2}{3} + k^2 = -19k + \frac{k^2}{16}$$

$$k = 48 \quad \text{or} \quad 0 \text{ (rejected)}$$

So, $h = \frac{2}{3} \times 48 = 32$.

27. D

Circumcentre is equidistant from the two vertices.

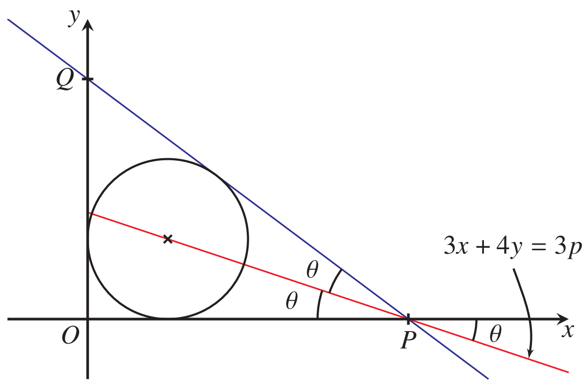
$$\sqrt{(k+4)^2 + (-4+8)^2} = \sqrt{(k-6)^2 + (-4-2)^2}$$

$$k^2 + 8k + 32 = k^2 - 12k + 72$$

$$k = 2$$

28. D

Note that $(p,\ 0)$ satisfies the equation $3x + 4y = 3p$.
The straight line $3x + 4y = 3p$ passes through $P\ (p,\ 0)$ and the in-centre of $\triangle OPQ$.
Therefore, it is the angle bisector of $\angle OPQ$.
Let the acute angle between the straight line and the x-axis be θ.



$$\text{Slope of the line} = -\frac{3}{4} = -\tan\theta \quad \text{and} \quad \frac{OQ}{OP} = \tan 2\theta$$

$$\theta = \tan^{-1}\frac{3}{4} \qquad \frac{q}{p} = \frac{24}{7}$$

$$p : q = 7 : 24$$

 END OF PAPER

29. D

The line $5x + 4y = 4b$ passes through $B\,(0,\ b)$.
Therefore, $5x + 4y = 4b$ passes through the midpoint of OA $\left(\frac{a}{2},\ 0\right)$.

$$5\left(\frac{a}{2}\right) + 0 = 4b$$
$$\frac{a}{b} = \frac{8}{5}$$

30. A

I. ✓. G lies inside $\triangle OAB$, which is in the second quadrant. The x- and y-coordinates are not equal (one positive and one negative).

II. ✓. Let the radius of inscribed circle be r. Then the coordinates of G are $(-r,\ r)$.

$$4r + (-r) = 3kb$$
$$r = kb$$

Using tangent properties, OB is divided into two segments with lengths $b - r$ and r.
OA is divided into two segments with lengths $10 - r$ and r.

$$(10 - r) + (b - r) = \sqrt{10^2 + b^2}$$
$$[10 + b(1 - 2k)]^2 = b^2 + 100$$
$$100 + 20b(1 - 2k) + b^2(1 - 2k)^2 = b^2 + 100$$
$$b^2(4k^2 - 4k) + 20b(1 - 2k) = 0$$
$$b = -\frac{20(1 - 2k)}{4k^2 - 4k}$$
$$= \frac{5(1 - 2k)}{k(1 - k)}$$

Required distance $= r = kb = \dfrac{5(1 - 2k)}{1 - k}$

III. ✗. When $k = \frac{1}{6}$, $r = \dfrac{5(1 - 2k)}{1 - k} = 4$.
Equation of inscribed circle is $(x + 4)^2 + (y - 4)^2 = 4^2$.

$$(x + 4)^2 + (5 - 3x - 4)^2 = 16$$
$$10x^2 + 2x + 1 = 0$$

$\Delta = 2^2 - 4(10)(1) = -36 < 0$.
The straight line $3x + y = 5$ does not cut the inscribed circle of $\triangle OAB$ and hence is not a tangent.