

M2REG-TRIG-2223-ASM-SET 3-MATH

Suggested solutions

$$\begin{aligned}
 1. \quad (a) \quad \tan B &= \tan\left(\frac{\pi}{4} - A\right) \\
 &= \frac{\tan \frac{\pi}{4} - \tan A}{1 + \tan \frac{\pi}{4} \tan A} \\
 &= \frac{1 - \frac{1}{1+2m}}{1 + 1 \cdot \frac{1}{1+2m}} && 1M \\
 &= \frac{(1+2m) - 1}{(1+2m) + 1} \\
 &= \frac{m}{1+m} && 1
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\
 &= \frac{\frac{1}{1+2m} - \frac{m}{1+m}}{1 + \left(\frac{1}{1+2m}\right)\left(\frac{m}{1+m}\right)} && 1M \\
 &= \frac{(1+m) - m(1+2m)}{(1+2m)(1+m)} \div \frac{(1+2m)(1+m) + m}{(1+2m)(1+m)} \\
 &= \frac{1 - 2m^2}{1 + 4m + 2m^2} && 1A
 \end{aligned}$$

$$\begin{aligned}
 2. \quad (a) \quad \cot(\angle A + \angle B) &= \frac{1}{\tan(\angle A + \angle B)} \\
 &= \frac{1 - \tan \angle A \tan \angle B}{\tan \angle A + \tan \angle B} && 1M \\
 &= \frac{1 - (\sqrt{a} + \sqrt{a-1})(\sqrt{a} - \sqrt{a-1})}{(\sqrt{a} + \sqrt{a-1}) + (\sqrt{a} - \sqrt{a-1})} \\
 &= \frac{1 - [a - (a-1)]}{2\sqrt{a}} \\
 &= 0 && 1A
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \cot(\angle A + \angle B) &= 0 \\
 \angle A + \angle B &= 90^\circ && 1A \\
 \angle C &= 180^\circ - (\angle A + \angle B) \\
 &= 90^\circ
 \end{aligned}$$

Thus, $\triangle ABC$ is a right-angled triangle. 1A

$$3. \quad (a) \quad k \cos(\alpha - \beta) = \cos(\alpha + \beta) \quad 1M$$

$$k(\cos \alpha \cos \beta + \sin \alpha \sin \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$(k + 1) \sin \alpha \sin \beta = (1 - k) \cos \alpha \cos \beta$$

$$(k + 1) \tan \alpha \tan \beta = 1 - k$$

$$\tan \alpha \tan \beta = \frac{1 - k}{1 + k} \quad 1$$

$$(b) \quad \frac{1 - k}{1 + k} = \tan(2\beta) \tan \beta$$

$$= \frac{2 \tan \beta}{1 - \tan^2 \beta} \cdot \tan \beta$$

$$= \frac{2 \tan^2 \beta}{1 - \tan^2 \beta} \quad 1A$$

$$(1 - k)(1 - \tan^2 \beta) = 2(1 + k) \tan^2 \beta$$

$$(3 + k) \tan^2 \beta = 1 - k$$

$$\tan^2 \beta = \frac{1 - k}{3 + k}$$

$$\tan \beta = \pm \sqrt{\frac{1 - k}{3 + k}} \quad 1A$$

Note that when $\pi < \alpha < 2\pi$, we have $\frac{\pi}{2} < \beta < \pi$ and $\tan \beta < 0$. 1A

Thus, $\tan \beta = -\sqrt{\frac{1 - k}{3 + k}}$. 1A

4. Since O is the orthocentre of $\triangle ABC$, we have $CD \perp AB$ and $BE \perp AC$.

$$\tan \alpha = \frac{CE}{BE}$$

$$= \frac{4\sqrt{6}}{6\sqrt{6}}$$

$$= \frac{2}{3} \quad 1A$$

$$\tan \beta = \frac{BD}{CD}$$

$$= \frac{3}{15}$$

$$= \frac{1}{5} \quad 1A$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{\frac{2}{3} + \frac{1}{5}}{1 - \frac{2}{3} \times \frac{1}{5}}$$

$$= 1 \quad 1A$$

Since $0 < \alpha + \beta < \pi$, we have $\alpha + \beta = \frac{\pi}{4}$. 1

$$\begin{aligned}
5. \quad & \tan(x+y) - \tan x - \tan y - \tan x \tan y \tan(x+y) \\
&= \tan(x+y)(1 - \tan x \tan y) - (\tan x + \tan y) \\
&= \frac{\tan x + \tan y}{1 - \tan x \tan y} (1 - \tan x \tan y) - (\tan x + \tan y) & 1\text{M}+1\text{M} \\
&= (\tan x + \tan y) - (\tan x + \tan y) \\
&= 0 \\
&\text{Thus, } \tan(x+y) - \tan x - \tan y = \tan x \tan y \tan(x+y). & 1
\end{aligned}$$

$$\begin{aligned}
6. \quad & \cos(x+y+z) = \cos(x+y) \cos z - \sin(x+y) \sin z & 1\text{M} \\
&= (\cos x \cos y - \sin x \sin y) \cos z - (\sin x \cos y + \cos x \sin y) \sin z & 1\text{M} \\
&= \cos x \cos y \cos z - \sin x \sin y \cos z - \sin x \cos y \sin z - \cos x \sin y \sin z \\
&= \cos x \cos y \cos z (1 - \tan x \tan y - \tan x \tan z - \tan y \tan z) & 1
\end{aligned}$$

$$\begin{aligned}
7. \quad & \frac{\cos(6\alpha + \beta)}{\sin 3\alpha} + 2 \sin(3\alpha + \beta) \\
&= \frac{\cos[(3\alpha + \beta) + 3\alpha] + 2 \sin(3\alpha + \beta) \sin 3\alpha}{\sin 3\alpha} \\
&= \frac{\cos(3\alpha + \beta) \cos 3\alpha - \sin(3\alpha + \beta) \sin 3\alpha + 2 \sin(3\alpha + \beta) \sin 3\alpha}{\sin 3\alpha} & 1\text{M} \\
&= \frac{\cos(3\alpha + \beta) \cos 3\alpha + \sin(3\alpha + \beta) \sin 3\alpha}{\sin 3\alpha} \\
&= \frac{\cos[(3\alpha + \beta) - 3\alpha]}{\sin 3\alpha} & 1\text{M} \\
&= \frac{\cos \beta}{\sin 3\alpha} & 1
\end{aligned}$$

$$\begin{aligned}
8. \quad & \frac{\cos A - \cos B \cos(A-B)}{\sin A - \cos B \sin(A-B)} \\
&= \frac{\cos A - \cos B(\cos A \cos B + \sin A \sin B)}{\sin A - \cos B(\sin A \cos B - \cos A \sin B)} & 1\text{M} \\
&= \frac{\cos A - \cos A \cos^2 B - \sin A \sin B \cos B}{\sin A - \sin A \cos^2 B + \cos A \sin B \cos B} \\
&= \frac{\cos A(1 - \cos^2 B) - \sin A \sin B \cos B}{\sin A(1 - \cos^2 B) + \cos A \sin B \cos B} \\
&= \frac{\cos A \sin^2 B - \sin A \sin B \cos B}{\sin A \sin^2 B + \cos A \sin B \cos B} \\
&= \frac{\sin B(\cos A \sin B - \sin A \cos B)}{\sin B(\sin A \sin B + \cos A \cos B)} \\
&= \frac{\sin(B-A)}{\cos(B-A)} & 1\text{M} \\
&= \tan(B-A) & 1
\end{aligned}$$

$$\begin{aligned}
 9. \quad (a) \quad \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} &= \frac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\cos \alpha \cos \beta + \sin \alpha \sin \beta} & 1M \\
 &= \frac{\frac{\cos \alpha \cos \beta}{\sin \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\sin \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\sin \alpha \cos \beta} + \frac{\sin \alpha \sin \beta}{\sin \alpha \cos \beta}} \\
 &= \frac{\cot \alpha - \tan \beta}{\cot \alpha + \tan \beta} & 1
 \end{aligned}$$

(b) We have $\cos(\alpha + \beta)(\cot \alpha + \tan \beta) = \cos(\alpha - \beta)(\cot \alpha - \tan \beta)$.

$$\begin{aligned}
 &\frac{1}{\cos 1^\circ} \sum_{i=1}^{90} \cos (2i - 1)^\circ [\cot i^\circ + \tan (i - 1)^\circ] \\
 &= \frac{1}{\cos 1^\circ} \sum_{i=1}^{90} \cos [i + (i - 1)]^\circ [\cot i^\circ + \tan (i - 1)^\circ] \\
 &= \frac{1}{\cos 1^\circ} \sum_{i=1}^{90} \cos [i - (i - 1)]^\circ [\cot i^\circ - \tan (i - 1)^\circ] & 1M \\
 &= (\cot 1^\circ + \cot 2^\circ + \cot 3^\circ + \dots + \cot 90^\circ) - (\tan 0^\circ + \tan 1^\circ + \tan 2^\circ + \dots + \tan 89^\circ) \\
 &= (\tan 89^\circ + \tan 88^\circ + \tan 87^\circ + \dots + \tan 0^\circ) - (\tan 0^\circ + \tan 1^\circ + \tan 2^\circ + \dots + \tan 89^\circ) & 1M \\
 &= 0 & 1A
 \end{aligned}$$

$$\begin{aligned}
 10. \quad &\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} \\
 &= \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{180^\circ - A - B}{2} + \tan \frac{180^\circ - A - B}{2} \tan \frac{A}{2} & 1M \\
 &= \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \cot \left(\frac{A}{2} + \frac{B}{2} \right) + \cot \left(\frac{A}{2} + \frac{B}{2} \right) \tan \frac{A}{2} \\
 &= \tan \frac{A}{2} \tan \frac{B}{2} + \left(\tan \frac{B}{2} + \tan \frac{A}{2} \right) \cdot \frac{1 - \tan \frac{A}{2} \tan \frac{B}{2}}{\tan \frac{A}{2} + \tan \frac{B}{2}} & 1M \\
 &= \tan \frac{A}{2} \tan \frac{B}{2} + 1 - \tan \frac{A}{2} \tan \frac{B}{2} \\
 &= 1 & 1
 \end{aligned}$$

$$\begin{aligned}
 11. \quad &\frac{\tan 10x - \tan 8x}{1 + \tan 10x \tan 8x} = \frac{\sqrt{3}}{3} \\
 &\tan(10x - 8x) = \frac{\sqrt{3}}{3} & 1M \\
 &\tan 2x = \frac{\sqrt{3}}{3} \\
 &2x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6} & 1M \\
 &x = \frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12} & 1A
 \end{aligned}$$

12. (a) $\tan(\alpha + \beta)(\tan \alpha + \tan \beta) + 2$

$$= \frac{(\tan \alpha + \tan \beta)^2}{1 - \tan \alpha \tan \beta} + 2 \quad 1\text{M}$$

$$= \frac{\tan^2 \alpha + 2 \tan \alpha \tan \beta + \tan^2 \beta + 2(1 - \tan \alpha \tan \beta)}{1 - \tan \alpha \tan \beta}$$

$$= \frac{\tan^2 \alpha + \tan^2 \beta + 2}{1 - \tan \alpha \tan \beta} \quad 1\text{M}$$

$$= \frac{(\tan^2 \alpha + 1) + (\tan^2 \beta + 1)}{1 - \tan \alpha \tan \beta}$$

$$= \frac{\sec^2 \alpha + \sec^2 \beta}{1 - \tan \alpha \tan \beta} \quad 1$$

(b) $\tan\left(\alpha + \frac{\pi}{4}\right)\left(\tan \alpha + \tan \frac{\pi}{4}\right) = -8$

$$\tan\left(\alpha + \frac{\pi}{4}\right)\left(\tan \alpha + \tan \frac{\pi}{4}\right) + 2 = -6$$

$$\frac{\sec^2 \alpha + \sec^2 \frac{\pi}{4}}{1 - \tan \alpha \tan \frac{\pi}{4}} = -6 \quad 1\text{M}$$

$$\frac{1 + \tan^2 \alpha + 2}{1 - \tan \alpha} = -6$$

$$\tan^2 \alpha + 3 = -6 + 6 \tan \alpha$$

$$\tan^2 \alpha - 6 \tan \alpha + 9 = 0 \quad 1\text{A}$$

$$\tan \alpha = 3$$

$$\alpha \approx 1.25 \quad \text{or} \quad 4.39 \quad 1\text{A}$$

13. (a) $\tan A$ and $\tan B$ are roots of the equation $x^2 + 6x - 3 = 0$.

We have $\tan A + \tan B = -6$ and $\tan A \tan B = -3$.

1A

$$(\tan A - \tan B)^2 = \tan^2 A - 2 \tan A \tan B + \tan^2 B$$

$$= (\tan A + \tan B)^2 - 4 \tan A \tan B$$

1M

$$= (-6)^2 - 4(-3)$$

$$= 48$$

1A

Since A is obtuse angle and B is acute angle, $\tan A < 0$ and $\tan B > 0$.

$$\tan A - \tan B = -\sqrt{48} = -4\sqrt{3}$$

1A

$$(b) \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

1M

$$= \frac{-4\sqrt{3}}{1 + (-3)}$$

$$= 2\sqrt{3}$$

1A

- (c) Since $\tan(A - B) > 0$, angle $A - B$ is an acute angle.

$$\sec(A - B) = \frac{\sqrt{(2\sqrt{3})^2 + 1}}{1}$$

$$= \sqrt{13}$$

1A

$$\csc(A - B) = \frac{\sqrt{13}}{2\sqrt{3}}$$

$$= \frac{\sqrt{39}}{6}$$

1A