M2REG-INVM-2223-ASM-SET 1-MATH
Suggested solutions

1. (a) $$|A-I|=\begin{vmatrix}0 & -5 & 5\\ 2 & 0 & -1\\ -2 & 1 & 0\end{vmatrix}$$

$$=-2\begin{vmatrix}-5 & 5\\ 0 & -1\end{vmatrix}-\begin{vmatrix}0 & 5\\ 2 & -1\end{vmatrix}$$ 1M

$$=0$$ 1A

(b) $\det(B-AB)=\det[-B(A-I)]$ 1M

$=\det(A-I)\det(-B)$ 1M

$=0$

Thus, $B-AB$ is a singular matrix. 1

2. $\det\left(I-A^{-1}BA\right)=\det\left(A^{-1}A-A^{-1}BA\right)$

$=\det\left[A^{-1}(I-B)A\right]$ 1M

$=\det A^{-1}\det(I-B)\det A$ 1M

$=\det\left(A^{-1}A\right)\det(I-B)$

$=\det I\det(I-B)$ 1M

$=\det(I-B)$ 1

3. (a) $\det\left(A^{-1}\right) = \det\left(A^T\right)$ 1M

$\dfrac{1}{\det A} = \det A$

$(\det A)^2 = 1$ 1

$\det(AB) = (\det A)(\det B)$

$= (\det A)(0 - \det A)$ 1M

$= -(\det A)^2$

$= -1$ 1

(b) $A(B + A)^T B = AB^T B + AA^T B$ 1M

$= AB^{-1}B + AA^{-1}B$ 1M

$= A + B$ 1

(c) $\det(A + B) = \det\left[A(B + A)^T B\right]$ 1M

$\det(A + B) = (\det A)[\det(B + A)](\det B)$

$\det(A + B) = (\det(AB))\det(A + B)$

$\det(A + B) = -\det(A + B)$ 1M

$2\det(A + B) = 0$

$\det(A + B) = 0$

Thus, $A + B$ is singular. 1

4. (a) $B\begin{pmatrix}-2 & -3 & -4\\ 2 & 4 & 5\\ -1 & -1 & -2\end{pmatrix}=\begin{pmatrix}-3 & -2 & 1\\ -1 & 0 & 2\\ 2 & 1 & -2\end{pmatrix}\begin{pmatrix}-2 & -3 & -4\\ 2 & 4 & 5\\ -1 & -1 & -2\end{pmatrix}$

$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ 1M

$\begin{pmatrix}-2 & -3 & -4\\ 2 & 4 & 5\\ -1 & -1 & -2\end{pmatrix}B=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Thus, $B^{-1}=\begin{pmatrix}-2 & -3 & -4\\ 2 & 4 & 5\\ -1 & -1 & -2\end{pmatrix}$. 1

(b) $(B^{-1})^2=\begin{pmatrix}-2 & -3 & -4\\ 2 & 4 & 5\\ -1 & -1 & -2\end{pmatrix}\begin{pmatrix}-2 & -3 & -4\\ 2 & 4 & 5\\ -1 & -1 & -2\end{pmatrix}$

$=\begin{pmatrix}2 & -2 & 1\\ -1 & 5 & 2\\ 2 & 1 & 3\end{pmatrix}$

$(B^{-1})^4=[(B^{-1})^2]^2$

$=\begin{pmatrix}2 & -2 & 1\\ -1 & 5 & 2\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}2 & -2 & 1\\ -1 & 5 & 2\\ 2 & 1 & 3\end{pmatrix}$

$=\begin{pmatrix}8 & -13 & 1\\ -3 & 29 & 15\\ 9 & 4 & 13\end{pmatrix}$

$(B^4)^{-1}=(B^{-1})^4$ 1M

$=\begin{pmatrix}8 & -13 & 1\\ -3 & 29 & 15\\ 9 & 4 & 13\end{pmatrix}$ 1A

5. (a) $XY = \begin{pmatrix} 2 & -1 & 5 \\ 0 & 2 & 1 \\ 1 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 & 22 & -11 \\ 1 & -1 & -2 \\ -2 & -9 & 4 \end{pmatrix}$

$= \begin{pmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{pmatrix}$ 1A

$YX = \begin{pmatrix} 0 & 22 & -11 \\ 1 & -1 & -2 \\ -2 & -9 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 & 5 \\ 0 & 2 & 1 \\ 1 & 4 & 2 \end{pmatrix}$

$= \begin{pmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{pmatrix}$ 1A

(b) Note that $X\left(-\frac{1}{11}Y\right) = \left(-\frac{1}{11}Y\right)X = I$. 1M

$X^{-1} = -\frac{1}{11}Y$

$= -\frac{1}{11}\begin{pmatrix} 0 & 22 & -11 \\ 1 & -1 & -2 \\ -2 & -9 & 4 \end{pmatrix}$ 1A

6. (a) $A^2 = \frac{1}{4}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$ 1A

$A^3 = \frac{1}{4}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 1A

(b) Note that $AA^2 = A^2A = I$. 1M

$A^{-1} = A^2$

$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$ 1A

7. (a) $\begin{vmatrix} 3 & -5 \\ 1 & -2 \end{vmatrix} = -1$

$$\begin{pmatrix} 3 & -5 \\ 1 & -2 \end{pmatrix}^{-1} = \frac{1}{-1}\begin{pmatrix} -2 & -1 \\ 5 & 3 \end{pmatrix}^{T} \qquad \text{1M}$$

$$= \begin{pmatrix} 2 & -5 \\ 1 & -3 \end{pmatrix} \qquad \text{1A}$$

(b) $\begin{pmatrix} 3 & -5 \\ 1 & -2 \end{pmatrix} A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \qquad \text{1M}$$

$$= \begin{pmatrix} 2 & 9 \\ 1 & 5 \end{pmatrix} \qquad \text{1A}$$

8. (a) $\det B = \begin{vmatrix} 12 & -5 \\ -21 & 8 \end{vmatrix} = -9$ 1A

$$B^{-1} = \frac{1}{-9}\begin{pmatrix} 8 & 5 \\ 21 & 12 \end{pmatrix} \qquad \text{1A}$$

(b) $\begin{pmatrix} 12 & -5 \\ -21 & 8 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 9 & 4 \\ 0 & 5 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} 8 & 5 \\ 21 & 12 \end{pmatrix}\begin{pmatrix} 9 & 4 \\ 0 & 5 \end{pmatrix} \qquad \text{1M}$$

$$= \begin{pmatrix} -8 & -\frac{19}{3} \\ -21 & -16 \end{pmatrix} \qquad \text{1A}$$

9. (a) $A^2 = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 9 \\ 6 & 7 \end{pmatrix}$ 1M

$$A^2 - 3A - 4I = \begin{pmatrix} 10 & 9 \\ 6 & 7 \end{pmatrix} - 3\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{1A}$$

(b) $A^2 - 3A - 4I = \mathbf{0}$

$$I = \frac{3}{4}A - \frac{1}{4}A^2$$

$$= A\left(\frac{3}{4}I - \frac{1}{4}A\right) = \left(\frac{3}{4}I - \frac{1}{4}A\right)A \qquad \text{1M}$$

$$A^{-1} = \frac{1}{4}(A - 3I) \qquad \text{1A}$$

$$= \begin{pmatrix} -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad \text{1A}$$

10. (a) $P^2 = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 1 \\ -4 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 1 \\ -4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 0 & -8 \\ 2 & 4 & 1 \\ -16 & 0 & 17 \end{pmatrix}$ 1A

$P^3 = \begin{pmatrix} 9 & 0 & -8 \\ 2 & 4 & 1 \\ -16 & 0 & 17 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 1 \\ -4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 41 & 0 & -42 \\ 6 & 8 & 3 \\ -84 & 0 & 83 \end{pmatrix}$ 1A

$P^3 - 6P^2 + 3P + 10I$

$= \begin{pmatrix} 41 & 0 & -42 \\ 6 & 8 & 3 \\ -84 & 0 & 83 \end{pmatrix} - 6\begin{pmatrix} 9 & 0 & -8 \\ 2 & 4 & 1 \\ -16 & 0 & 17 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 1 \\ -4 & 0 & 3 \end{pmatrix} + 10\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 1

(b) $P^3 - 6P^2 + 3P + 10I = \mathbf{0}$

$I = \frac{1}{10}(-P^3 + 6P^2 - 3P)$

$= P\left[-\frac{1}{10}P^2 + \frac{3}{5}P - \frac{3}{10}I\right] = \left[-\frac{1}{10}P^2 + \frac{3}{5}P - \frac{3}{10}I\right]P$ 1M

$P^{-1} = -\frac{1}{10}P^2 + \frac{3}{5}P - \frac{3}{10}I$ 1A

$= \begin{pmatrix} -\frac{3}{5} & 0 & -\frac{2}{5} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ -\frac{4}{5} & 0 & -\frac{1}{5} \end{pmatrix}$ 1A

11. (a) $M^2 = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}$ 1A

$$M^3 + M = \frac{1}{3\sqrt{3}}\begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}$$

$$= -\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$= -M$$

Thus, $M^3 + M = \mathbf{0}$. 1

(b) $\det M = -\frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$

$= 0$ 1A

$\det\left(M^3\right) = (\det M)^3$

$= 0$ 1A

(c) $(I + M + M^2)(I - M + M^2) = I - M + M^2 + M - M^2 + M^3 + M^2 - M^3 + M^4$ 1M

$= I + M^2 + M^4$

$= I + M^2 + M(M^3)$

$= I + M^2 + M(-M)$ 1M

$= I$

$(I - M + M^2)(I + M + M^2) = I + M + M^2 - M - M^2 - M^3 + M^2 + M^3 + M^4$

$= I + M^2 + M^4$

$= I$

Thus, $(I + M + M^2)^{-1} = I - M + M^2$. 1

(d) $(I + M)X = M^2(I - X)$

$(I + M + M^2)X = M^2$ 1M

$X = (I - M + M^2)M^2$ 1M

$= M^2 - M^3 + M^4$

$= M^2 + M + M(-M)$ 1M

$= M$

$= \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ 1A

12. (a) $\det M = \begin{vmatrix} m & m \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2m \\ 2 & 1 \end{vmatrix}$

$= -m - (-4m)$

$= 3m$

$M^{-1} = \frac{1}{3m}\begin{pmatrix} -m & -1 & 2 \\ 4m & 1 & -2 \\ -2m^2 & m & m \end{pmatrix}^T$ 1M

$= \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} & -\frac{2m}{3} \\ -\frac{1}{3m} & \frac{1}{3m} & \frac{1}{3} \\ \frac{2}{3m} & -\frac{2}{3m} & \frac{1}{3} \end{pmatrix}$ 1A

(b) $M\begin{pmatrix} x \\ y \\ -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} & -\frac{2m}{3} \\ -\frac{1}{3m} & \frac{1}{3m} & \frac{1}{3} \\ \frac{2}{3m} & -\frac{2}{3m} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ 1M

$= \begin{pmatrix} \frac{2m+8}{3} \\ \frac{-m-1}{3m} \\ \frac{-m+2}{3m} \end{pmatrix}$

$-\frac{1}{4} = \frac{-m+2}{3m}$ 1M

$-3m = -4m + 8$

$m = 8$ 1A

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