Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Extended Part

S5 – S6 M2 Integration Assignment $\boxed{\text{Set 7}}$

Name: ____________________ Centre: __________

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings must be clearly shown.
3. Unless otherwise specified, numerical answers must be exact.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution

Distributed in summer course
S5 – S6 M2 Integration
Phase 2 – Lesson 3

1. (a) (i) $\cos\theta(2\cos 4\theta - 2\cos 2\theta + 1) = 2\cos\theta\cos 4\theta - 2\cos\theta\cos 2\theta + \cos\theta$

$= (\cos 5\theta + \cos 3\theta) - (\cos 3\theta + \cos\theta) + \cos\theta$ 1M

$= \cos 5\theta$

Thus, $\dfrac{\cos 5\theta}{\cos\theta} = 2\cos 4\theta - 2\cos 2\theta + 1$. 1

(ii) $\dfrac{\cos 5\left(\frac{\pi}{4} - t\right)}{\cos\left(\frac{\pi}{4} - t\right)} = 2\cos\left(\dfrac{\pi}{4} - t\right) - 2\cos 2\left(\dfrac{\pi}{4} - t\right) + 1$

$\dfrac{\cos\frac{5\pi}{4}\cos 5t + \sin\frac{5\pi}{4}\sin 5t}{\cos\frac{\pi}{4}\cos t + \sin\frac{\pi}{4}\sin t} = -2\cos 4t - 2\sin 2t + 1$ 1M

$\dfrac{-\frac{\sqrt{2}}{2}\cos 5t - \frac{\sqrt{2}}{2}\sin 5t}{\frac{\sqrt{2}}{2}\cos t + \frac{\sqrt{2}}{2}\sin t} = -2\cos 4t - 2\sin 2t + 1$

$\dfrac{\cos 5t + \sin 5t}{\cos t + \sin t} = 2\cos 4t + 2\sin 2t - 1$ 1

(b) (i) Let $u = a - x$. Then $\mathrm{d}u = -\mathrm{d}x$. 1M

$\int_0^a f(a-x)\,\mathrm{d}x = -\int_a^0 f(u)\,\mathrm{d}u$

$= \int_0^a f(u)\,\mathrm{d}u$ 1M

$= \int_0^a f(x)\,\mathrm{d}x$ 1

(ii) $\int_0^{\frac{\pi}{2}} \dfrac{\cos 5x}{\cos x + \sin x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \dfrac{\cos 5\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)}$ 1M

$= \int_0^{\frac{\pi}{2}} \dfrac{\sin 5x}{\cos x + \sin x}\,\mathrm{d}x$ 1

(c) $\int_0^{\frac{\pi}{2}} \dfrac{\cos 5x}{\cos x + \sin x}\,\mathrm{d}x$

$= \dfrac{1}{2}\left[\int_0^{\frac{\pi}{2}} \dfrac{\cos 5x}{\cos x + \sin x}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \dfrac{\sin 5x}{\cos x + \sin x}\,\mathrm{d}x\right]$ 1M

$= \dfrac{1}{2}\int_0^{\frac{\pi}{2}} (2\cos 4x + 2\sin 2x - 1)\,\mathrm{d}x$ 1M

$= \dfrac{1}{2}\left[\dfrac{\sin 4x}{2} - \cos 2x - x\right]_0^{\frac{\pi}{2}}$

$= 1 - \dfrac{\pi}{4}$ 1A

2. (a) Let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_{-a}^{0} f(x)\,\mathrm{d}x = -\int_{a}^{0} f(-u)\,\mathrm{d}u$$

$$= \int_{0}^{a} f(-u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_{0}^{a} f(-x)\,\mathrm{d}x \qquad \text{1}$$

(b) $$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x$$

$$= \int_{0}^{a} f(-x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_{0}^{a} [f(x) + f(-x)]\,\mathrm{d}x \qquad \text{1}$$

(c) Put $f(x) = \ln\left(\sqrt{e^6 + x^6} - x^3\right) + \cos\dfrac{\pi x}{2}$.

$f(x) + f(-x)$

$$= \left[\ln\left(\sqrt{e^6 + x^6} - x^3\right) + \cos\frac{\pi x}{2}\right] + \left[\ln\left(\sqrt{e^6 + (-x)^6} - (-x)^3\right) + \cos\frac{\pi(-x)}{2}\right]$$

$$= \ln[(\sqrt{e^6 + x^6} - x^3)(\sqrt{e^6 + x^6} + x^3)] + 2\cos\frac{\pi x}{2}$$

$$= \ln[(e^6 + x^6) - x^6] + 2\cos\frac{\pi x}{2}$$

$$= 6 + 2\cos\frac{\pi x}{2} \qquad \text{1}$$

$$\int_{-3}^{3} \left[\ln\left(\sqrt{e^6 + x^6} - x^3\right) + \cos\frac{\pi x}{2}\right]\mathrm{d}x$$

$$= \int_{0}^{3} [f(x) + f(-x)]\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_{0}^{3} \left(6 + 2\cos\frac{\pi x}{2}\right)\mathrm{d}x$$

$$= \left[6x + \frac{4}{\pi}\sin\frac{\pi x}{2}\right]_{0}^{3}$$

$$= 18 - \frac{4}{\pi} \qquad \text{1A}$$

3. (a) Let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x \qquad \text{1M}$$

$$= -\int_{a}^{0} f(-u)\,\mathrm{d}u + \int_{0}^{a} f(x)\,\mathrm{d}x$$

$$= \int_{0}^{a} f(-u)\,\mathrm{d}u + \int_{0}^{a} f(x)\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_{0}^{a} f(-x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x$$

$$= \int_{0}^{a} [f(x) + f(-x)]\,\mathrm{d}x \qquad \text{1}$$

(b) $$\int_{-\pi}^{\pi} \ln\left(\sqrt{e^3 + \sin^2 x} - \sin x\right)\mathrm{d}x$$

$$= \int_{0}^{\pi} \left[\ln\left(\sqrt{e^3 + \sin^2 x} - \sin x\right) + \ln\left(\sqrt{e^3 + (-\sin x)^2} - (-\sin x)\right)\right]\mathrm{d}x \qquad \text{1M}$$

$$= \int_{0}^{\pi} \left[\ln\left(\sqrt{e^3 + \sin^2 x} - \sin x\right) + \ln\left(\sqrt{e^3 + \sin^2 x} + \sin x\right)\right]\mathrm{d}x \qquad \text{1}$$

(c) $$\int_{-\pi}^{\pi} \ln\left(\sqrt{e^3 + \sin^2 x} - \sin x\right)\mathrm{d}x$$

$$= \int_{0}^{\pi} \left[\ln\left(\sqrt{e^3 + \sin^2 x} - \sin x\right) + \ln\left(\sqrt{e^3 + \sin^2 x} + \sin x\right)\right]\mathrm{d}x$$

$$= \int_{0}^{\pi} \ln\left[\left(\sqrt{e^3 + \sin^2 x} - \sin x\right)\left(\sqrt{e^3 + \sin^2 x} + \sin x\right)\right]\mathrm{d}x \qquad \text{1M}$$

$$= \int_{0}^{\pi} \ln\left[\left(e^3 + \sin^2 x\right) - \sin^2 x\right]\mathrm{d}x$$

$$= \int_{0}^{\pi} \ln e^3\,\mathrm{d}x$$

$$= 3\int_{0}^{\pi} \mathrm{d}x \qquad \text{1A}$$

$$= 3\Big[x\Big]_{0}^{\pi}$$

$$= 3\pi \qquad \text{1A}$$

(d) Let $u = x - \pi$. Then $\mathrm{d}x = \mathrm{d}x$. 1M

$$\int_{0}^{2\pi} \ln\left(e^3 + 2\sin^2 x + 2\sin x\sqrt{e^3 + \sin^2 x}\right)\mathrm{d}x$$

$$= \int_{-\pi}^{\pi} \ln\left(e^3 + 2\sin^2(\pi + u) + 2\sin(\pi + u)\sqrt{e^3 + \sin^2(\pi + u)}\right)\mathrm{d}u$$

$$= \int_{-\pi}^{\pi} \ln\left((e^3 + \sin^2 u) - 2\sin u\sqrt{e^3 + \sin^2 u} + \sin^2 u\right)\mathrm{d}u \qquad \text{1M}$$

$$= \int_{-\pi}^{\pi} \ln\left(\sqrt{e^3 + \sin^2 u} - \sin u\right)^2\mathrm{d}x$$

$$= 2\int_{-\pi}^{\pi} \ln\left(\sqrt{e^3 + \sin^2 u} - \sin u\right)\mathrm{d}u \qquad \text{1M}$$

$$= 2(3\pi)$$

$$= 6\pi \qquad \text{1A}$$

4. (a) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\begin{aligned}\frac{1}{2}\int_0^a [f(x)+f(a-x)]\,\mathrm{d}x &= \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{2}\int_0^a f(a-x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x - \frac{1}{2}\int_a^0 f(u)\,\mathrm{d}u \\ &= \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{2}\int_0^a f(u)\,\mathrm{d}u &&\text{1M} \\ &= \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x \\ &= \int_0^a f(x)\,\mathrm{d}x &&\text{1}\end{aligned}$$

(b) $x^2 - 6x + 18 = (x-3)^2 + 9$

Let $x - 3 = 3\tan\theta$. Then $\mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int_0^6 \frac{\mathrm{d}x}{x^2-6x+18} &= \int_0^6 \frac{\mathrm{d}x}{(x-3)^2+9} \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{3\sec^2\theta}{9\tan^2\theta+9}\,\mathrm{d}\theta &&\text{1M} \\ &= \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{d}\theta &&\text{1A} \\ &= \frac{1}{3}\Big[\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= \frac{\pi}{6} &&\text{1A}\end{aligned}$$

(c) Put $f(x) = \dfrac{1}{(x^2-6x+18)(e^{3-x}+1)}$.

$f(x) + f(6-x)$

$$\begin{aligned}&= \frac{1}{(x^2-6x+18)(e^{3-x}+1)} + \frac{1}{[(6-x)^2-6(6-x)+18](e^{3-(6-x)}+1)} \\ &= \frac{1}{(x^2-6x+18)(e^{3-x}+1)} + \frac{1}{(x^2-6x+18)(e^{-(3-x)}+1)} \\ &= \frac{1}{(x^2-6x+18)(e^{3-x}+1)} + \frac{e^{3-x}}{(x^2-6x+18)(1+e^{3-x})} \\ &= \frac{1}{x^2-6x+18} &&\text{1A}\end{aligned}$$

$$\begin{aligned}&\int_0^6 \frac{\mathrm{d}x}{(x^2-6x+18)(e^{3-x}+1)} \\ &= \frac{1}{2}\int_0^6 [f(x)+f(6-x)]\,\mathrm{d}x &&\text{1M} \\ &= \frac{1}{2}\int_0^6 \frac{\mathrm{d}x}{x^2-6x+18} \\ &= \frac{1}{2}\left(\frac{\pi}{6}\right) &&\text{1M} \\ &= \frac{\pi}{12} &&\text{1A}\end{aligned}$$

5. (a) Let $u = x^3$. Then $du = 3x^2\,dx$. 1M

$$\int x^2 \sin x^3\,dx = \frac{1}{3}\int \sin u\,du$$
$$= -\frac{\cos u}{3} + \text{constant}$$
$$= -\frac{\cos x^3}{3} + \text{constant}$$ 1A

(i) Let $u = a + b - x$. Then $du = -dx$. 1M

$$\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,dx$$
$$= -\int_b^a \frac{f(a+b-u)}{f(u)+f(a+b-u)}\,du$$
$$= \int_a^b \frac{f(a+b-u)}{f(u)+f(a+b-u)}\,du$$ 1M
$$= \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}\,dx$$ 1A

$$\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,dx$$
$$= \frac{1}{2}\left[\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,dx + \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx\right]$$ 1M
$$= \frac{1}{2}\int_a^b dx$$ 1A
$$= \frac{1}{2}\Big[x\Big]_a^b$$
$$= \frac{b-a}{2}$$ 1

(ii) Let $u = x^3$. Then $du = 3x^2\,dx$. 1M

$$\int_{\sqrt[3]{\log 5}}^{\sqrt[3]{\log 6}} \frac{x^2 \sin x^3}{\sin(\log 30 - x^3) + \sin x^3}\,dx$$
$$= \frac{1}{3}\int_{\log 5}^{\log 6} \frac{\sin u}{\sin(\log 5 + \log 6 - u) + \sin u}\,du$$ 1M
$$= \frac{1}{3}\cdot\frac{\log 6 - \log 5}{2}$$ 1M
$$= \frac{1}{6}\log\frac{6}{5}$$ 1A

6. (a) Let $u = a + b - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_a^b f(a+b-x)\,\mathrm{d}x = -\int_b^a f(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_a^b f(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_a^b f(x)\,\mathrm{d}x \qquad \text{1}$$

(b) $$\int_0^{\frac{\pi}{4}} \ln\sin 2x\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\sin\left[2\left(\frac{\pi}{4} - x\right)\right]\mathrm{d}x \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{4}} \ln\cos 2x\,\mathrm{d}x \qquad \text{1}$$

(c) Let $2x = t$. Then $2\,\mathrm{d}x = \mathrm{d}t$. 1M

$$\int_0^{\frac{\pi}{4}} \ln\sin 4x\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin 2t\,\mathrm{d}t$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln\sin 2t\,\mathrm{d}t + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\sin 2t\,\mathrm{d}t\right] \qquad \text{1M}$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln\sin 2t\,\mathrm{d}t + \int_0^{\frac{\pi}{4}} \ln\sin\left[2\left(u + \frac{\pi}{4}\right)\right]\mathrm{d}u\right] \quad \text{where } u = t - \frac{\pi}{4} \qquad \text{1M}$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln\sin 2t\,\mathrm{d}t + \int_0^{\frac{\pi}{4}} \ln\cos 2u\,\mathrm{d}u\right]$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln\sin 2x\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln\sin 2x\,\mathrm{d}x\right]$$

$$= \int_0^{\frac{\pi}{4}} \ln\sin 2x\,\mathrm{d}x \qquad \text{1}$$

(d) $$\int_0^{\frac{\pi}{4}} \ln\sin 4x\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} (\ln 2 + \ln\sin 2x + \ln\cos 2x)\,\mathrm{d}x$$

$$= \ln 2\Big[x\Big]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \ln\sin 2x\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln\cos 2x\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi\ln 2}{4} + 2\int_0^{\frac{\pi}{4}} \ln\sin 4x\,\mathrm{d}x \qquad \text{1M}$$

$$\int_0^{\frac{\pi}{4}} \ln\sin 4x\,\mathrm{d}x = -\frac{\pi\ln 2}{4} \qquad \text{1A}$$