Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Extended Part

S5 – S6 M2 Integration Assignment $\boxed{\text{Set 5}}$

Name: ______________________________ Centre: ______________

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings must be clearly shown.
3. Unless otherwise specified, numerical answers must be exact.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution

Distributed in summer course
S5 – S6 M2 Integration
Phase 2 – Lesson 1

1. (a) $\dfrac{A}{x+1}+\dfrac{B}{x+3}=\dfrac{A(x+3)+B(x+1)}{(x+1)(x+3)}$

$$=\frac{(A+B)x+(3A+B)}{(x+1)(x+3)}$$

We have $A+B=1$ and $3A+B=0$. 1M

Solving, we have $A=-\dfrac{1}{2}$ and $B=\dfrac{3}{2}$. 1A

(b) $\displaystyle\int \frac{x}{(x+1)(x+3)}\,\mathrm{d}x=\int\left(-\frac{1}{2(x+1)}+\frac{3}{2(x+3)}\right)\mathrm{d}x$ 1M

$$=-\frac{1}{2}\ln|x+1|+\frac{3}{2}\ln|x+3|+\text{constant}$$ 1A

(c) Let $u=\sqrt{x}$. Then $\mathrm{d}u=\dfrac{1}{2\sqrt{x}}\,\mathrm{d}x$. 1M

$$\int_1^9 \frac{1}{(\sqrt{x}+1)(\sqrt{x}+3)}\,\mathrm{d}x=2\int_1^9\frac{\sqrt{x}}{2\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+3)}\,\mathrm{d}x$$

$$=2\int_1^3\frac{u}{(u+1)(u+3)}\,\mathrm{d}u$$

$$=2\left[-\frac{1}{2}\ln|u+1|+\frac{3}{2}\ln|u+3|\right]_1^3$$ 1M

$$=3\ln 3-4\ln 2$$ 1A

2. (a) $\dfrac{x}{x^2+4}-\dfrac{1}{x+4}=\dfrac{x(x+4)-(x^2+4)}{(x^2+4)(x+4)}$ 1M

$=\dfrac{4x-4}{(x^2+4)(x+4)}$

$=\dfrac{4(x-1)}{(x^2+4)(x+4)}$ 1

(b) Let $t=2\tan\theta$. Then $\mathrm{d}t=2\sec^2\theta\,\mathrm{d}\theta$.

$$\int_0^{\frac{\pi}{4}}\frac{2\tan\theta-1}{\tan\theta+2}\,\mathrm{d}\theta=\int_0^{\frac{\pi}{4}}\frac{2\tan\theta-1}{2\sec^2\theta(\tan\theta+2)}\cdot 2\sec^2\theta\,\mathrm{d}\theta$$

$$=\int_0^2\frac{t-1}{2\left(1+\frac{t^2}{4}\right)\left(\frac{t}{2}+2\right)}\,\mathrm{d}t$$ 1A

$$=\int_0^2\frac{4(t-1)}{(t^2+4)(t+4)}\,\mathrm{d}t$$

$$=\int_0^2\left(\frac{t}{t^2+4}-\frac{1}{t+4}\right)\mathrm{d}t$$ 1M

Let $u=t^2+4$. Then $\mathrm{d}u=2t\,\mathrm{d}t$. 1M

$$\int_0^{\frac{\pi}{4}}\frac{2\tan\theta-1}{\tan\theta+2}\,\mathrm{d}\theta=\frac{1}{2}\int_4^8\frac{\mathrm{d}u}{u}-\int_0^2\frac{\mathrm{d}t}{t+4}$$

$$=\frac{1}{2}\Big[\ln|u|\Big]_4^8-\Big[\ln|t+4|\Big]_0^2$$ 1A

$$=\frac{1}{2}(\ln 8-\ln 4)-(\ln 6-\ln 2)$$

$$=\frac{5}{2}\ln 2-\ln 6$$ 1A

3. (a) $\displaystyle\int_4^2\frac{2f(x)}{f(x)-2g(x)}\,\mathrm{d}x=-2\int_2^4\frac{f(x)}{f(x)-2g(x)}\,\mathrm{d}x$

$=-2(10)$

$=-20$ 1A

(b) $\displaystyle\int_2^4\frac{g(t)}{f(t)-2g(t)}\,\mathrm{d}t=\int_2^4\frac{g(x)}{f(x)-2g(x)}\,\mathrm{d}x$

$$=-\frac{1}{2}\int_2^4\left[1-\frac{f(x)}{f(x)-2g(x)}\right]\mathrm{d}x$$ 1M

$$=-\frac{1}{2}\Big[x\Big]_2^4+\frac{1}{2}\int_2^4\frac{f(x)}{f(x)-2g(x)}\,\mathrm{d}x$$

$$=-\frac{1}{2}(4-2)+\frac{1}{2}(10)$$

$=4$ 1A

4. (a) $\dfrac{dy}{dx} = \tan^2 x \sec^2 x - \sec^2 x + 1$ 1M

$= \tan^2 x(1 + \tan^2 x) - \tan^2 x$

$= \tan^4 x$ 1A

(b) $\displaystyle\int_0^{\frac{\pi}{6}} 3\tan^4 x\,dx = 3\int_0^{\frac{\pi}{6}} \tan^4 x\,dx$

$= 3\left[\dfrac{1}{3}\tan^3 4x - \tan x + x\right]_0^{\frac{\pi}{6}}$ 1M

$= \dfrac{\pi}{2} - \dfrac{8\sqrt{3}}{9}$ 1A

5. (a) Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$.

$\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$ 1M

$= \displaystyle\int d\theta$

$= \sin^{-1} x + \text{constant}$ 1A

(b) $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \sqrt{\frac{1-x}{1+x}}\,dx = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \sqrt{\frac{(1-x)^2}{(1+x)(1-x)}}\,dx$ 1M

$= \displaystyle\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \left(\frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}\right)dx$

$= \displaystyle\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2}\int_{\frac{3}{4}}^{\frac{1}{2}} \frac{du}{\sqrt{u}}$ where $u = 1 - x^2$ 1M

$= \left[\sin^{-1} x\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} + \dfrac{1}{2}\left[2u^{\frac{1}{2}}\right]_{\frac{3}{4}}^{\frac{1}{2}}$ 1M

$= \dfrac{\pi}{12} + \dfrac{\sqrt{2} - \sqrt{3}}{2}$ 1A

6. (a) Let $u = 1 - e^{-x}$. Then $du = e^{-x}\,dx$. 1M

$$\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}}\,dx = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{du}{u}$$ 1A

$$= \Big[\ln|u|\Big]_{\frac{1}{2}}^{\frac{1}{3}}$$

$$= \ln\frac{4}{3}$$ 1A

(b) $$\frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1) + Bu}{u(u-1)}$$

$$= \frac{(A+B)u - A}{u(u-1)}$$

We have $A + B = 0$ and $-A = 1$. 1M

Solving, we have $A = -1$ and $B = 1$. 1A

(c) $$\int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x - 1)}\,dx = \int_{\ln 2}^{\ln 3} \left(-\frac{1}{e^x} + \frac{1}{e^x - 1}\right) dx$$ 1M

$$= -\int_{\ln 2}^{\ln 3} e^{-x}\,dx + \int_{\ln 2}^{\ln 3} \frac{1}{e^x - 1}\,dx$$

$$= -\int_{\ln 2}^{\ln 3} e^{-x}\,dx + \int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}}\,dx$$

$$= \Big[e^{-x}\Big]_{\ln 2}^{\ln 3} + \ln\frac{4}{3}$$ 1M

$$= -\frac{1}{6} + \ln\frac{4}{3}$$ 1A

7. (a) $$\frac{dy}{dx} = 2xe^{\frac{1}{x}} + x^2 e^{x}\left(-\frac{1}{x^2}\right)$$ 1M

$$= (2x - 1)e^{\frac{1}{x}}$$ 1A

(b) $$\int_1^2 \left[(2x-1)e^{\frac{1}{x}} - e^x\right] dx = \int_1^2 (2x-1)e^{\frac{1}{x}}\,dx - \int_1^2 e^x\,dx$$

$$= \Big[x^2 e^{\frac{1}{x}}\Big]_1^2 - \Big[e^x\Big]_1^2$$ 1M

$$= 4e^{\frac{1}{2}} - e^2$$ 1A

8. $$\int_0^2 \frac{x^3 - 27}{x - 3}\,dx = \int_0^2 \frac{(x-3)(x^2 + 3x + 9)}{x - 3}\,dx$$ 1M

$$= \int_0^2 (x^2 + 3x + 9)\,dx$$

$$= \left[\frac{x^3}{3} + \frac{3x^2}{2} + 9x\right]_0^2$$ 1M

$$= \frac{80}{3}$$ 1A

9. (a) $2^3 - 6(2)^2 + 2k - 4 = 0$ 1M

$k = 10$ 1A

(b) $f(x) = x^3 - 6x^2 + 10x - 4$

$= (x-2)(x^2 - 4x + 2)$ 1A

(c) $\int_3^6 \frac{f(x)}{x-2}\,dx = \int_3^6 (x^2 - 4x + 2)\,dx$ 1M

$= \left[\frac{x^3}{3} - 2x^2 + 2x\right]_3^6$ 1M

$= 15$ 1A

10. $\int_0^{\frac{\pi}{3}} \frac{\sin^2 3x}{1+\cos 2x}\,dx + \int_{\frac{\pi}{3}}^0 \frac{\sin^2 3x - 2}{1+\cos 2x}\,dx = \int_0^{\frac{\pi}{3}} \frac{\sin^2 3x}{1+\cos 2x}\,dx + \int_0^{\frac{\pi}{3}} \frac{2-\sin^2 3x}{1+\cos 2x}\,dx$ 1M

$= 2\int_0^{\frac{\pi}{3}} \frac{dx}{1+\cos 2x}\,dx$

$= 2\int_0^{\frac{\pi}{3}} \frac{dx}{2\cos^2 x}\,dx$

$= \int_0^{\frac{\pi}{3}} \sec^2 x\,dx$ 1A

$= \left[\tan x\right]_0^{\frac{\pi}{3}}$ 1M

$= \sqrt{3}$ 1A

11. (a) $y = 5^{x+1}$

$= e^{(x+1)\ln 5}$ 1M

$\frac{dy}{dx} = e^{(x+1)\ln 5}(\ln 5)$ 1M

$= 5^{x+1}\ln 5$ 1A

(b) $\int_1^3 5^{x+1}\,dx = \left[\frac{5^{x+1}}{\ln 5}\right]_1^3$ 1M

$= \frac{600}{\ln 5}$ 1A

12. Let $u = x^2 + 3$. Then $du = 2x\,dx$. 1M

$\int_1^3 \frac{x^2 - 2x + 3}{x^2 + 3}\,dx = \int_1^3 \left(1 - \frac{2x}{x^2+3}\right)dx$ 1M

$= \int_1^3 dx - \int_4^{12} \frac{du}{u}$ 1A

$= \left[x\right]_1^3 - \left[\ln|u|\right]_4^{12}$ 1M

$= 2 - \ln 3$ 1A

13. Let $u = 2x + 1$. Then $\mathrm{d}u = 2\,\mathrm{d}x$. 1M

$$\int_0^4 \frac{x+2}{\sqrt{2x+1}}\,\mathrm{d}x = \frac{1}{2}\int_1^9 \frac{\frac{u-1}{2}+2}{\sqrt{u}}\,\mathrm{d}u$$

$$= \frac{1}{2}\int_1^9 \left(\frac{\sqrt{u}}{2} + \frac{3}{2\sqrt{u}}\right)\mathrm{d}u \qquad \text{1M}$$

$$= \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{3} + 3u^{\frac{1}{2}}\right]_1^9$$

$$= \frac{22}{3} \qquad \text{1A}$$