Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Extended Part

S5 – S6 M2 Differentiation Assignment Set 8

Name: ____________________ Centre: __________

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings must be clearly shown.
3. Unless otherwise specified, numerical answers must be exact.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution

Distributed in summer course
S5 – S6 M2 Differentiation
Phase 2 – Lesson 4

1. (a) Note that $\angle ORQ = \dfrac{\theta}{2}$.

$$\cot\frac{\theta}{2} = \frac{RQ}{OQ} \qquad \text{1M}$$

$$= \frac{z^2 + 10}{z} \qquad \text{1A}$$

$$= z + \frac{10}{z}$$

(b)

$$\tan\frac{\theta}{2} = \frac{z}{z^2 + 10}$$

$$\frac{1}{2}\sec^2\frac{\theta}{2}\frac{d\theta}{dt} = \frac{(z^2 + 10) - z(2z)}{(z^2 + 10)^2} \cdot \frac{dz}{dt} \qquad \text{1M}$$

When $PQ = 4$, $z = 2$ and $\tan\dfrac{\theta}{2} = \dfrac{2}{2^2 + 10} = \dfrac{1}{7}$.

$$\frac{1}{2}\left[1 + \left(\frac{1}{7}\right)^2\right]\frac{d\theta}{dt} = \frac{(4 + 10) - 8}{(14)^2} \cdot (5) \qquad \text{1M}$$

$$\frac{d\theta}{dt} = \frac{3}{10}$$

$$\angle RSO = \frac{\pi - \theta}{2}$$

$$\frac{d}{dt}(\angle RSO) = -\frac{1}{2}\frac{d\theta}{dt}$$

$$= -\frac{3}{20}$$

Required rate is $-\dfrac{3}{20}$ rad/s. 1A

2. (a) (i) $27\pi = \pi r^2 h$ 1M

$$h = \frac{27}{r^2}$$

$$A = 2\pi rh + \pi r^2$$

$$= 2\pi r \times \frac{27}{r^2} + \pi r^2$$ 1M

$$= \pi\left(\frac{54}{r} + r^2\right)$$ 1

(ii) $\frac{dA}{dr} = \pi\left(-\frac{54}{r^2} + 2r\right)$

When $\frac{dA}{dr} = 0$,

$$2r = \frac{54}{r^2}$$

$$r = 3$$ 1A

r	$0 < r < 3$	$r > 3$
$\frac{dA}{dr}$	$-$	$+$

1M

A attains its minimum when $r = 3$. 1A

(b) (i) $F = \frac{1}{3}\pi(3)^2(5) - \frac{1}{3}\pi(3)^2(5) \times \left(\frac{5-D}{5}\right)^3$ 1M

$$= 15\pi\left[1 - \left(\frac{5-D}{5}\right)^3\right]$$ 1

(ii) Let V cm^3 the volume of water when the depth of water is D cm.

$$\pi(3)^2 D = V + F$$ 1M

$$9\pi D = V + 15\pi\left[1 - \left(\frac{5-D}{5}\right)^3\right]$$

$$9\pi\frac{dD}{dt} = \frac{dV}{dt} + \frac{9\pi}{25}(5-D)^2\frac{dD}{dt}$$ 1M

When the water inside the can is just full, $D = \frac{27\pi}{3^2\pi} = 3$.

$$9\pi\frac{dD}{dt} = \pi + \frac{9\pi}{25}(5-3)^2\frac{dD}{dt}$$ 1M

$$\frac{dD}{dt} = \frac{25}{189}$$

Required rate is $\frac{25}{189}$ cm/s. 1A

3. (a) Coordinates of A and B are $(0,\ 8)$ and $(2,\ 8)$ respectively. 1A+1A

(b) P lies on C, so $q = -2p^2 + 4p + 8$.

Area of $\triangle ABP = \frac{1}{2}(2)[8 - (-2p^2 + 4p + 8)]$ 1M

$= 2p^2 - 4p$ 1A

(c) $\frac{dq}{dt} = -4p\frac{dp}{dt} + 4\frac{dp}{dt}$ 1A

When $p = 4$,

$-8 = -4(4)\frac{dp}{dt} + 4\frac{dp}{dt}$ 1M

$\frac{dp}{dt} = \frac{2}{3}$

Let the area of $\triangle ABP$ be A.

$\frac{dA}{dt} = (4p - 4)\frac{dp}{dt}$ 1A

When $p = 4$,

$\frac{dA}{dt} = (16 - 4) \cdot \frac{2}{3}$

$= 8$

The rate of change of the area of $\triangle ABP$ is 8 square units per second. 1A

4. (a) Let r cm be the radius of the water surface in the container.

$\frac{r}{h} = \tan 30°$

$r = \frac{h}{\sqrt{3}}$ 1M

$V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$

$= \frac{\pi}{9}h^3$ 1

$A = \pi r\sqrt{r^2 + h^2}$

$= \pi \cdot \frac{h}{\sqrt{3}}\left(\frac{2h}{\sqrt{3}}\right)$

$= \frac{2\pi}{3}h^2$ 1

(b) When $V = \frac{64\pi}{9}$,

$\frac{\pi}{9}h^3 = \frac{64\pi}{9}$

$h = 4$ 1A

When $h = 4$,

$V = \frac{\pi}{9}h^3$

$\frac{dV}{dt} = \frac{\pi}{9}(3h^2)\frac{dh}{dt}$ 1M

$8\pi = \frac{\pi}{9}(48)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{3}{2}$

Also,

$\frac{dA}{dt} = \left(\frac{4\pi}{3}h\right)\frac{dh}{dt}$

$= \frac{16\pi}{3}\left(\frac{3}{2}\right)$

$= 8\pi$

Required rate is 8π cm^2/s. 1A

5. (a) $\ell^2 = (OC + OP\cos\theta)^2 + (OB + OP\sin\theta)^2$

$= (1+\cos\theta)^2 + (1+\sin\theta)^2$ 1A+1A

$= 2 + (\sin^2\theta + \cos^2\theta) + 2\sin\theta + 2\cos\theta$

$= 3 + 2(\sin\theta + \cos\theta)$ 1

(b) $\dfrac{d\ell^2}{dt} = 2(\cos\theta - \sin\theta)\dfrac{d\theta}{dt}$

$2\ell\dfrac{d\ell}{dt} = 2(\cos\theta - \sin\theta)\dfrac{d\theta}{dt}$ 1M

$\dfrac{d\ell}{dt} = \dfrac{\cos\theta - \sin\theta}{\ell}\dfrac{d\theta}{dt}$ 1A

When $\theta = \dfrac{\pi}{3}$,

$\ell^2 = 3 + 2\left(\sin\dfrac{\pi}{3} + \cos\dfrac{\pi}{3}\right)$

$\ell = \sqrt{4+\sqrt{3}}$ 1A

$\dfrac{d\ell}{dt} = \dfrac{\cos\frac{\pi}{3} - \sin\frac{\pi}{3}}{\ell}(3)$

≈ -0.46

Required rate is -0.46 cm/s. 1A

6. (a) Volume of the pyramid $= \frac{1}{3}(12)^2(6) = 288\,\text{cm}^3$.

Volume of pyramid below water surface

$= 288\left[1 - \left(\frac{6-h}{6}\right)^3\right]$ 1M

$= \frac{4}{3}[6-(6-h)][6^2+6(6-h)+(6-h)^2]$

$= \frac{4h^3}{3} - 24h^2 + 144h$

$V = 12h^2 - \left(\frac{4h^3}{3} - 24h^2 + 144h\right)$ 1M

$= 24h^2 - \frac{4}{3}h^3$ 1

(b) $\frac{\mathrm{d}V}{\mathrm{d}t} = 48h\frac{\mathrm{d}h}{\mathrm{d}t} - 4h^2\frac{\mathrm{d}h}{\mathrm{d}t}$ 1M

Put $\frac{\mathrm{d}h}{\mathrm{d}t} = 1$,

$\frac{\mathrm{d}V}{\mathrm{d}t} = 48h - 4h^2 > 140$

$-4h^2 + 48h - 140 > 0$ 1M

$5 < h < 7$ 1M

$h \le 5$ by considering the height of the container. It is not possible to have $\frac{\mathrm{d}V}{\mathrm{d}t} > 140$.

The claim is disagreed. 1A

(c) Put $\frac{\mathrm{d}V}{\mathrm{d}t} = -(12-h)(\ln h + 1)^2$,

$-(12-h)(\ln h+1)^2 = (48h - 4h^2)\frac{\mathrm{d}h}{\mathrm{d}t}$ 1M

$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{(\ln h + 1)^2}{4h}$

$\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = \frac{2(\ln h+1)\left(\frac{1}{h}\right)(h) - (\ln h+1)^2}{-4h^2}$ 1M

$= \frac{(\ln h+1)(\ln h-1)}{4h^2}$

When $\dfrac{d^2h}{dt^2} = 0$,

$\ln h = \pm 1$

$h = e$ or $\dfrac{1}{e}$

h	$0 < h < \frac{1}{e}$	$\frac{1}{e} < h < e$	$e < h < 5$
$\frac{d^2h}{dt^2}$	$+$	$-$	$+$

1M

$\left.\dfrac{dh}{dt}\right|_{h=\frac{1}{e}} = 0$ and $\left.\dfrac{dh}{dt}\right|_{h=e} = -\dfrac{1}{e}$ and $\left.\dfrac{dh}{dt}\right|_{h=5} = -\dfrac{(\ln 5+1)^2}{20}$. When depth of water decreases from 5 cm to e cm, the rate of change of the depth of water decreases from $-\dfrac{(\ln 5+1)^2}{20}$ cm/s to a local minimum $-\dfrac{1}{e}$ cm/s. 1M

When the depth of water decreases from e cm to $\dfrac{1}{e}$ cm, the rate of change of the depth of water increases from $-\dfrac{1}{e}$ cm/s to a local maximum 0 cm/s.

When the depth of water decreases from $\dfrac{1}{e}$ cm to 0 cm, the rate of change of the depth of water decreases indefinitely. 1A