Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Extended Part

S5 – S6 M2 Differentiation Assignment $\boxed{\text{Set 7}}$

Name: ____________________ Centre: __________

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

**INSTRUCTIONS**

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings must be clearly shown.
3. Unless otherwise specified, numerical answers must be exact.
4. The diagrams in this paper are not necessarily drawn in scale.

**Suggested solution**

Distributed in summer course
S5 – S6 M2 Differentiation
Phase 2 – Lesson 3

1. (a) Let $T$ hours be the time taken.

$$T = \frac{x}{8} + \frac{\sqrt{(6-x)^2+4^2}}{3}$$ 1A

$$= \frac{x}{8} + \frac{\sqrt{x^2-12x+52}}{3}$$

(b) $$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{1}{8} + \frac{1}{6}(x^2-12x+52)^{-\frac{1}{2}}(2x-12)$$ 1M

$$= \frac{1}{8} + \frac{2x-12}{6(x^2-12x+52)^{\frac{1}{2}}}$$

When $\frac{\mathrm{d}T}{\mathrm{d}x} = 0$,

$$\frac{1}{8} + \frac{2x-12}{6\sqrt{x^2-12x+52}} = 0$$ 1M

$$(8x-48)^2 = 3^2(x^2-12x+52)$$

$$55x^2 - 660x + 1836 = 0$$

$$x = \frac{330-12\sqrt{55}}{55} \quad \text{or} \quad \frac{330+12\sqrt{55}}{55} \text{ (rejected)}$$ 1A

| $x$ | $0 < x < \frac{330-12\sqrt{55}}{55}$ | $\frac{330-12\sqrt{55}}{55} < x < 6$ |
|---|---|---|
| $\frac{\mathrm{d}T}{\mathrm{d}x}$ | $-$ | $+$ |

1M

$T$ attains minimum when $x = \frac{330-12\sqrt{55}}{55} \approx 4.38$.

Minimum time taken $= \frac{x}{8} + \frac{\sqrt{x^2-12x+52}}{3}$

$\approx 1.99\text{ h} > 1.5\text{ h}$ 1A

The claim is disagreed. 1A

2. (a) $32\pi = 2\pi rh + 2\pi r^2$ 1M

$h = \dfrac{16}{r} - r$ 1

(b) $V = \pi r^2 h + \dfrac{2}{3}\pi r^3$

$= \pi r^2\left(\dfrac{16}{r} - r\right) + \dfrac{2}{3}\pi r^3$ 1M

$= \left(16r - \dfrac{1}{3}r^3\right)\pi$ 1

(c) $\dfrac{\mathrm{d}V}{\mathrm{d}r} = (16 - r^2)\pi$

When $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 0$, $r = 4$ or $-4$ (rejected) 1M

Since the outer surface area is $32\pi\ \mathrm{cm}^2$, we have $0 < r \le 4$.

As $\dfrac{\mathrm{d}V}{\mathrm{d}r} > 0$ for $0 < r < 4$, $V$ attains its maximum at $r = 4$. 1M

Maximum capacity $= \left(16(4) - \dfrac{1}{3}(4)^3\right)\pi = \dfrac{128\pi}{3}\ \mathrm{cm}^3 < 135\ \mathrm{cm}^3$. 1A

The capacity cannot be greater than $135\ \mathrm{cm}^3$. 1A

3. (a) $\dfrac{x}{\sin\left(\pi - \frac{2\pi}{3} - \theta\right)} = \dfrac{3}{\sin\frac{2\pi}{3}}$ 1M

$$x = \frac{3\sin\left(\frac{\pi}{3} - \theta\right)}{\left(\frac{\sqrt{3}}{2}\right)}$$

$= 2\sqrt{3}\sin\left(\dfrac{\pi}{3} - \theta\right)$ 1A

(b) $y = \dfrac{1}{2}(x)(3)\sin\theta$

$= 3\sqrt{3}\sin\left(\dfrac{\pi}{3} - \theta\right)\sin\theta$ 1

(c) $y = \dfrac{3\sqrt{3}}{2}\left[\cos\left(\dfrac{\pi}{3} - 2\theta\right) - \cos\dfrac{\pi}{3}\right]$

$\dfrac{dy}{d\theta} = 3\sqrt{3}\sin\left(\dfrac{\pi}{3} - 2\theta\right)$ 1

(d) When $\dfrac{dy}{d\theta} = 0$,

$$\sin\left(\frac{\pi}{3} - 2\theta\right) = 0$$

$$\frac{\pi}{3} - 2\theta = 0$$

$\theta = \dfrac{\pi}{6}$ 1A

| $\theta$ | $0 < \theta < \frac{\pi}{6}$ | $\frac{\pi}{6} < \theta < \frac{\pi}{3}$ |
|---|---|---|
| $\frac{dy}{d\theta}$ | + | − |

1M

$y$ attains its greatest value at $\theta = \dfrac{\pi}{6}$.

Required value $= 3\sqrt{3}\sin\dfrac{\pi}{6}\sin\dfrac{\pi}{6} = \dfrac{3\sqrt{3}}{4}$ 1A

4. (a) (i) $AE = AC - CE$

$1\cot\theta = 8\cos\theta - x$ 1M

$x = 8\cos\theta - \cot\theta$ 1A

(ii) $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \csc^2\theta - 8\sin\theta$ 1A

When $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 0$,

$$\frac{1}{\sin^2\theta} = 8\sin\theta$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

| $\theta$ | $0 < \theta < \dfrac{\pi}{6}$ | $\dfrac{\pi}{6} < \theta < \dfrac{\pi}{2}$ |
|---|---|---|
| $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ | + | − |

1M

$x$ attains its maximum when $\theta = \dfrac{\pi}{6}$. 1

(b) (i) $S = \dfrac{1}{2}(8\sin\theta)(8\cos\theta) - \dfrac{1}{2}(1\cot\theta)(1)$ 1M

$= \dfrac{1}{2}(32\sin 2\theta - \cot\theta)$ 1

(ii) $\dfrac{\mathrm{d}S}{\mathrm{d}\theta} = 32\cos 2\theta + \dfrac{1}{2}\csc^2\theta$ 1A

When $\theta = \dfrac{\pi}{6}$, $\dfrac{\mathrm{d}S}{\mathrm{d}\theta} = 18 \neq 0$. 1M

Thus, $S$ does not attain maximum when $x$ attains its maximum, i.e., when $\theta = \dfrac{\pi}{6}$. 1A

5. (a) (i) $\dfrac{FG}{BC} = \dfrac{6-x}{6}$ 1M

$FG = \dfrac{2\sqrt{3}(6-x)}{3}$ cm

$S = \dfrac{1}{2}\left(\dfrac{2\sqrt{3}(6-x)}{3}\right)(x)$

$= \dfrac{\sqrt{3}x(6-x)}{3}$ 1

(ii) $\dfrac{dS}{dx} = \dfrac{\sqrt{3}}{3}(6-2x)$ 1A

When $\dfrac{dS}{dx} = 0$, $x = 3$. 1M

| $x$ | $0 < x < 3$ | $3 < x < 6$ |
|---|---|---|
| $\dfrac{dS}{dx}$ | $+$ | $-$ |

1M

$S$ attains its maximum when $x = 3$. 1A

(b) (i) $P = \dfrac{2\sqrt{3}(6-x)}{3} + 2\sqrt{x^2 + \left[\dfrac{\sqrt{3}(6-x)}{3}\right]^2}$ 1M

$= \dfrac{2\sqrt{3}(6-x)}{3} + 2\sqrt{x^2 + \dfrac{x^2 - 12x + 36}{3}}$

$= \dfrac{2\sqrt{3}(6-x)}{3} + 4\sqrt{\dfrac{x^2 - 3x + 9}{3}}$ 1A

(ii) $\dfrac{dP}{dx} = -\dfrac{2\sqrt{3}}{3} + \dfrac{2}{\sqrt{3}}(x^2 - 3x + 9)^{-\frac{1}{2}}(2x - 3)$

$= -\dfrac{2\sqrt{3}}{3} + \dfrac{2(2x-3)}{\sqrt{3(x^2 - 3x + 9)}}$ 1A

When $\dfrac{dP}{dx} = 0$,

$$\dfrac{2(2x-3)}{\sqrt{3(x^2 - 3x + 9)}} = \dfrac{2\sqrt{3}}{3}$$

$$(2x-3)^2 = x^2 - 3x + 9$$

$$3x^2 - 9x = 0$$

$$x = 3 \quad \text{or} \quad 0 \text{ (rejected)}$$

| $x$ | $0 < x < 3$ | $3 < x < 6$ |
|---|---|---|
| $\dfrac{dP}{dx}$ | $-$ | $+$ |

1M

$P$ attains its minimum when $S$ attains its maximum at $x = 3$.

The claim is disagreed. 1A

6. (a) $100e^{-2.5} = [-(25)^2 + A(25) + B]e^{-\frac{25}{10}}$ 1M

$25A + B = 725$

$75e^{-3} = [-(30)^2 + A(30) + B]e^{-\frac{30}{10}}$

$30A + B = 975$

Solving, we have $A = 50$ and $B = -525$. 1A+1A

(b) (i) $\frac{\mathrm{d}W(x)}{\mathrm{d}x} = (-x^2 + 50x - 525)e^{-\frac{x}{10}}\left(-\frac{1}{10}\right) + e^{-\frac{x}{10}}(-2x + 50)$ 1M

$= \frac{1}{10}(x^2 - 70x + 1025)e^{-\frac{x}{10}}$ 1A

(ii) When $\frac{\mathrm{d}W(x)}{\mathrm{d}x} = 0$,

$x^2 - 70x + 1025 = 0$ 1M

$x = 35 - 10\sqrt{2}$ or $35 + 10\sqrt{2}$ (rejected) 1A

| $x$ | $15 < x < 35 - 10\sqrt{2}$ | $35 - 10\sqrt{2} < x < 35$ |
|---|---|---|
| $\frac{\mathrm{d}W(x)}{\mathrm{d}x}$ | $+$ | $-$ |

1M

$W(x)$ attains its maximum at $x = 35 - 10\sqrt{2}$.

Maximum weight $= [-(35 - 10\sqrt{2})^2 + 50(35 - 10\sqrt{2}) - 525]e^{-\frac{35-10\sqrt{2}}{10}}$

$\approx 10.29\,\mathrm{t}$ 1A

(c) $P(x) = \frac{1000W(x)}{x}$

$\frac{\mathrm{d}P(x)}{\mathrm{d}x} = 1000\left[\frac{xW'(x) - W(x)}{x^2}\right]$ 1M

$\left.\frac{\mathrm{d}P(x)}{\mathrm{d}x}\right|_{x=35-10\sqrt{2}} = 1000\left[\frac{(35 - 10\sqrt{2})W'(35 - 10\sqrt{2}) - W(35 - 10\sqrt{2})}{(35 - 10\sqrt{2})^2}\right]$

$= 1000\left[\frac{-W(35 - 10\sqrt{2})}{(35 - 10\sqrt{2})^2}\right]$

$\neq 0$ 1M

$P(x)$ does not attains its maximum when $W(x)$ attains its maximum.

The claim is incorrect. 1A

7. $f'(x) = \dfrac{1 - \ln x}{x^2}$ 1A

When $f'(x) = 0$,

$$1 - \ln x = 0$$ 1M

$$x = e$$

| $x$ | 1 | $e$ | 4 |
|---|---|---|---|
| $f(x)$ | 0 | $\dfrac{1}{e}$ | $\dfrac{\ln 4}{4}$ |

1M

Since $0 < \dfrac{\ln 4}{4} < \dfrac{1}{e}$,

global maximum $= \dfrac{1}{e}$ and global minimum $= 0$. 1A+1A

8. (a) (i) $a = \dfrac{(AB)(1)}{2} + \dfrac{(BC)(1)}{2} + \dfrac{(CA)(1)}{2}$ 1M

$2a = AB + BC + CA$

$s = 2a$ 1

(ii) $AB + BC + CA = 2a$

$x + x + 2x\sin\theta = 2 \times \dfrac{1}{2}(x)^2 \sin 2\theta$ 1M+1M

$2x(1+\sin\theta) = x^2 \sin 2\theta$

$x = \dfrac{2(1+\sin\theta)}{\sin 2\theta}$

$s = 2 \times \dfrac{2(1+\sin\theta)}{\sin 2\theta} \times (1+\sin\theta)$

$= \dfrac{4(1+\sin\theta)^2}{\sin 2\theta}$ 1

(b) $\dfrac{ds}{d\theta} = \dfrac{8(1+\sin\theta)(\cos\theta)(\sin 2\theta) - 4(1+\sin\theta)^2(2\cos 2\theta)}{\sin^2 2\theta}$ 1M

$= \dfrac{8(1+\sin\theta)(\cos\theta\sin 2\theta - \sin\theta\cos 2\theta - \cos 2\theta)}{\sin^2 2\theta}$

$= \dfrac{8(1+\sin\theta)[\sin(2\theta-\theta) - \cos 2\theta]}{\sin^2 2\theta}$

$= \dfrac{8(1+\sin\theta)(\sin\theta - \cos 2\theta)}{\sin^2 2\theta}$ 1A

When $\dfrac{ds}{d\theta} = 0$, $\sin\theta = -1$ (rejected) or $\sin\theta - \cos 2\theta = 0$.

$\sin\theta - (1 - 2\sin^2\theta) = 0$

$2\sin^2\theta + \sin\theta - 1 = 0$

$\sin\theta = \dfrac{1}{2}$ or $-1$ (rejected)

$\theta = \dfrac{\pi}{6}$ 1A

| $\theta$ | $0 < \theta < \dfrac{\pi}{6}$ | $\dfrac{\pi}{6} < \theta < \dfrac{\pi}{2}$ |
|---|---|---|
| $\dfrac{ds}{d\theta}$ | $-$ | $+$ |

1M

Perimeter is minimum when $\theta = \dfrac{\pi}{6}$. 1A

(c) $x = \dfrac{2(1+\sin\theta)}{\sin 2\theta}$

$\dfrac{dx}{d\theta} = \dfrac{2\cos\theta\sin 2\theta - 2(1+\sin\theta)(2\cos 2\theta)}{\sin^2 2\theta}$ 1M

When $\theta = \dfrac{\pi}{6}$,

$$\frac{ds}{d\theta} = \frac{2\sin\frac{\pi}{3}\cos\frac{\pi}{6} - 4\cos\frac{\pi}{3}\left(1+\sin\frac{\pi}{6}\right)}{\sin^2\frac{\pi}{3}} = -2 \neq 0$$ 1M

The value of $x$ does not attain its minimum.

The claim is disagreed. 1A

9. (a) $N = 2PC + PA$ 1M

$= 2h\sec\theta + (50 - h\tan\theta)$ 1

(b) $N = 100\sec\theta + 50 - 50\tan\theta$

$\dfrac{dN}{d\theta} = 100\sec\theta\tan\theta - 50\sec^2\theta$ 1A

$= 50\sec^2\theta(2\sin\theta - 1)$

When $\dfrac{dN}{d\theta} = 0$,

$\sin\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{6}$ 1A

| $\theta$ | $0 < \theta < \frac{\pi}{6}$ | $\frac{\pi}{6} < \theta < \frac{\pi}{2}$ |
|---|---|---|
| $\frac{dN}{d\theta}$ | $-$ | $+$ |

1M

$N$ attains its minimum when $\theta = \dfrac{\pi}{6}$.

Required cost $= 50\sec\dfrac{\pi}{6}\left(2\tan\dfrac{\pi}{6} - \sec\dfrac{\pi}{6}\right) = \$50(\sqrt{3}+1)$ 1A

(c) (i) $\tan\theta = \dfrac{BP}{BC} < \dfrac{AB}{BC} = \dfrac{1}{\sqrt{3}}$ 1

When $\tan\theta < \dfrac{1}{\sqrt{3}}$, $0 < \theta < \dfrac{\pi}{6}$ and so $\dfrac{dN}{d\theta} < 0$ by (a). 1

(ii) Since $h = 100 > 50\sqrt{3}$, we have $\dfrac{dN}{d\theta} < 0$. 1M

$N$ is minimum when $\theta$ is maximum, which means the truck goes to $A$ directly. 1A



**END OF PAPER**