Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Compulsory Part

S4 – S5 Core Assignment Set 7

Name: ____________________ Centre: __________

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings (except for multiple choice questions) must be clearly shown.
3. Unless otherwise specified, numerical answers should be either exact or correct to 3 significant figures.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution

Distributed in summer course
S4 – S5 Core
Phase 2 – Lesson 3

1. B

Coordinates of centre are $(1,\ 0)$.

Slope of radius connecting $A = \dfrac{0-1}{1-0}$

$= -1$

Slope of tangent = 1

Required equation is $y = x + 1$.

2. A

Let the equation of L be $y = mx$ such that L passes through the origin.

$$x^2 + (mx)^2 - x + 5(mx) + 2 = 0$$
$$(1 + m^2)x^2 + (5m - 1)x + 2 = 0$$

$$\Delta = (5m-1)^2 - 4(1+m^2)(2) = 0$$
$$17m^2 - 10m - 7 = 0$$
$$m = 1 \quad \text{or} \quad -\frac{7}{17}$$

We have $m = 1$.

$$(1 + 1^2)x^2 + (5 - 1)x + 2 = 0$$
$$2x^2 + 4x + 2 = 0$$
$$x = -1$$

When $x = -1$, $y = (1)(-1) = -1$.

The coordinates of P are $(-1,\ -1)$.

3. D

$$(0)^2 + y^2 + 4(0) - 4 = 0$$
$$y^2 - 4 = 0$$
$$y = 2 \quad \text{or} \quad -2 \text{ (rejected)}$$
$$(0) - (2) + k = 0$$
$$k = 2$$
$$x^2 + (x+2)^2 + 4x - 4 = 0$$
$$2x^2 + 8x = 0$$
$$x = 0 \quad \text{or} \quad -4$$

When $x = -4$, $y = (-4) + 2 = -2$.

Coordinates of B are $(-4,\ -2)$.

4. A

$$x^2 + (2x+1)^2 - x + 2(2x+1) = 0$$
$$5x^2 + 7x + 3 = 0$$
$$\Delta = 7^2 - 4(5)(3) = -11 < 0$$

Required number is 0.

5. B

Since $\angle AOB = 90°$, AB is a diameter of the circle.

Coordinates of centre are $\left(-\frac{5}{2}, -3\right)$.

Slope of radius connecting the origin $= \dfrac{-3-0}{-\frac{5}{2}-0}$

$= \dfrac{6}{5}$

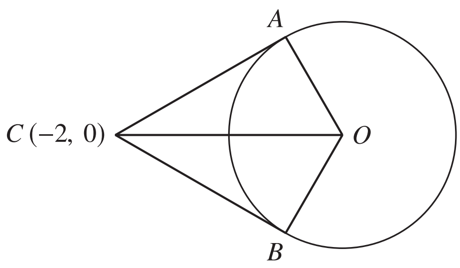
Slope of tangent $= -\dfrac{5}{6}$

Required equation is $5x + 6y = 0$.

6. D

Radius of circle = 1; $\angle AOC = \cos^{-1}\dfrac{1}{2} = 60°$

$\angle AOB = 120°$ and $AB = \sqrt{1^2 + 1^2 - 2(1)(1)\cos 120°} = \sqrt{3}$



7. C

$(x-1)^2 + (0+1)^2 = 5$

$(x-1)^2 = 4$

$x - 1 = \pm 2$

$x = 3 \quad \text{or} \quad -1$

$AB = 3 - (-1) = 4$

Radius of circle $= 2\sqrt{5}$

$BC = \sqrt{(2\sqrt{5})^2 - (4)^2} = 2$

Required area = $(4)(2) = 8$

8. D

$x^2 + (x+4)^2 - 4x - 16 = 0$

$2x^2 + 4x = 0$

$x = 0 \quad \text{or} \quad -2$

When $x = 0$, $y = 0 + 4 = 4$; when $x = -2$, $y = -2 + 4 = 2$.

Required coordinates are $(0, 4)$ and $(-2, 2)$.

9. (a) Slope of $L_2 = \frac{3}{4}$ 1M

So, slope of $L_1 = -\frac{4}{3}$.

The coordinates of the centre of C are $(5,\ 0)$. 1M

The equation of L_1 is

$$y - 0 = -\frac{4}{3}(x - 5)$$ 1M

$$4x + 3y - 20 = 0$$ 1A

(b) $$x^2 + \left(-\frac{4x}{3} + \frac{20}{3}\right)^2 - 10x = 0$$ 1M

$$\left(1 + \frac{16}{9}\right)x^2 + \left(-\frac{160}{9} - 10\right)x + \frac{400}{9} = 0$$

$$\frac{25}{9}x^2 - \frac{250}{9}x + \frac{400}{9} = 0$$ 1M

$$x = 8 \quad \text{or} \quad 2$$

When $x = 8$, $y = -\frac{4}{3}(8) + \frac{20}{3} = -4$; when $x = 2$, $y = 4$.

The coordinates of the intersections are $(8,\ -4)$ and $(2,\ 4)$. 1A

10. (a) $$x^2 + \left(\frac{2x}{3}\right)^2 - 12x - 8\left(\frac{2x}{3}\right) + 39 = 0$$ 1M

$$\left(1 + \frac{4}{9}\right)x^2 + \left(-12 - \frac{16}{3}\right)x + 39 = 0$$

$$\frac{13}{9}x^2 - \frac{52}{3}x + 39 = 0$$ 1M

$$x = 3 \quad \text{or} \quad 9$$

When $x = 3$, $y = 2$; when $x = 9$, $y = 6$.

The coordinates of P and Q are $(3,\ 2)$ and $(9,\ 6)$ respectively. 1A+1A

(b) (i) Denote the centre of C_1 by G. Then coordinates of G are $(6,\ 4)$. 1M

Slope of $GP = \frac{4-2}{6-3} = \frac{2}{3}$.

So, slope of the common tangent is $-\frac{3}{2}$. 1M

The equation of the common tangent is

$$y - 2 = -\frac{3}{2}(x - 3)$$

$$3x + 2y - 13 = 0$$ 1A

(ii) The slope of L is $\frac{2}{3}$ and is perpendicular to the tangent to C_1 at P.

Therefore, L passes through the centre of C_2.

So, centre of C_2 is the midpoint of OP, whose coordinates are $\left(\frac{3}{2},\ 1\right)$.

The equation of C_2 is

$$\left(x - \frac{3}{2}\right)^2 + (y - 1)^2 = \left(0 - \frac{3}{2}\right)^2 + (0 - 1)^2$$ 1M

$$x^2 + y^2 - 3x - 2y = 0$$ 1A

11. (a) Coordinates of centre are $(-4,\ 0)$.

Slope of $CA = \dfrac{0+4}{-4+7} = \dfrac{4}{3}$. So, slope of tangent at $A = -\dfrac{3}{4}$. 1M

The equation of tangent at A is

$$y + 4 = -\frac{3}{4}(x+7)$$

$$3x + 4y + 37 = 0$$ 1A

As CB is vertical, tangent at B is horizontal. 1M

The equation of tangent at B is $y = 5$. 1A

(b) $3x + 4(5) + 37 = 0$ 1M

$$x = -19$$

Coordinates of D are $(-19,\ 5)$. 1A

(c) $\angle CAD = 90°$ *(tangent ⊥ radius)*

$\angle CBD = 90°$ *(tangent ⊥ radius)*

$\angle CAD + \angle CBD = 90° + 90° = 180°$ 1M

Thus, A, B, C and D are concylcic. *(opp. ∠s supp.)* 1A

12. (a) $$x^2 + (1-x)^2 - 2x + 6(1-x) + 1 = 0$$ 1M

$$(1+1)x^2 + (-2-2-6)x + (1+6+1) = 0$$

$$2x^2 - 10x + 8 = 0$$ 1M

$$x = 1 \quad \text{or} \quad 4$$

When $x = 1$, $y = 1 - 1 = 0$; when $x = 4$, $y = -3$

The coordinates of A and B are $(1,\ 0)$ and $(4,\ -3)$ respectively. 1A+1A

(b) The smallest circle has AB as a diameter.

Coordinates of centre $= \left(\dfrac{1+4}{2},\ \dfrac{0-3}{2}\right) = \left(\dfrac{5}{2},\ \dfrac{-3}{2}\right)$ 1M

The equation of the circle is

$$\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \left(1 - \frac{5}{2}\right)^2 + \left(0 + \frac{3}{2}\right)^2$$ 1M

$$\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{9}{2}$$ 1A

13. (a) $x^2 + \left(\frac{1}{3}x\right)^2 - 4x - 4\left(\frac{1}{3}x\right) = 0$ 1M

$$\frac{10}{9}x^2 - \frac{16}{3}x = 0$$

$$x = 0 \quad \text{or} \quad \frac{24}{5}$$

When $x = \frac{24}{5}$, $y = \frac{8}{5}$. The coordinates of P are $\left(\frac{24}{5}, \frac{8}{5}\right)$. 1A

Length of $OP = \sqrt{\left(\frac{24}{5}\right)^2 + \left(\frac{8}{5}\right)^2} = \frac{8\sqrt{10}}{5}$ 1A

(b) Let the coordinates of Q be (a, b). Since $PQ = OP$,

$$\left(a - \frac{24}{5}\right)^2 + \left(b - \frac{8}{5}\right)^2 = \left(\frac{8\sqrt{10}}{5}\right)^2 \quad \text{1M}$$

$$a^2 + b^2 - \frac{48}{5}a - \frac{16}{5}b = 0$$

Since (a, b) lies on the circle, $a^2 + b^2 - 4a - 4b = 0$.

$$\left(-\frac{48}{5}a - \frac{16}{5}b\right) - (-4a - 4b) = 0 \quad \text{1M}$$

$$-\frac{28}{5}a + \frac{4b}{5} = 0$$

$$b = 7a$$

Substitute $b = 7a$ into $a^2 + b^2 - 4a - 4b = 0$,

$$a^2 + (7a)^2 - 4a - 4(7a) = 0$$

$$50a^2 - 32a = 0$$

$$a = \frac{16}{25} \quad \text{or} \quad 0$$

When $a = \frac{16}{25}$, $b = \frac{112}{25}$. The coordinates of Q are $\left(\frac{16}{25}, \frac{112}{25}\right)$. 1A

The coordinates of the centre of circle are $(2, 2)$.

Slope of $CQ = \frac{\frac{112}{25} - 2}{\frac{16}{25} - 2} = -\frac{31}{17}$. 1M

Slope of tangent to the circle at $Q = \frac{17}{31}$. 1M

The equation of tangent to the circle at Q is

$$y - \frac{112}{25} = \frac{17}{31}\left(x - \frac{16}{25}\right) \quad \text{1M}$$

$$775y - 3472 = 425x - 272$$

$$17x - 31y + 128 = 0 \quad \text{1A}$$

14. (a) Substitute $(-2,\ 1)$,

$$\text{LHS} = (-2)^2 + 1^2 - 10(1) + 5 \qquad \text{1M}$$
$$= 0 = \text{RHS}$$

Thus, $P\ (-2,\ 1)$ lies on the circle. 1

(b) Coordinates of centre C are $(0,\ 5)$.

Slope of $CP = \dfrac{5-1}{0+2} = 2$

So, slope of the tangent $L = -\dfrac{1}{2}$. 1M

The equation of tangent is

$$y - 1 = -\frac{1}{2}(x+2)$$
$$x + 2y = 0 \qquad \text{1A}$$

(c) Let the point of contact of L_1 and S be A.

If L_1 is parallel to L, then C is the midpoint of AP.

Coordinates of A are $(2,\ 9)$. 1M

The equation of L_1 is

$$y - 9 = -\frac{1}{2}(x-2) \qquad \text{1M}$$
$$x + 2y - 20 = 0 \qquad \text{1A}$$

15. (a)

$$x^2 + \left(-\frac{3x}{4} + \frac{17}{2}\right)^2 - 6x - 16 = 0 \qquad \text{1M}$$
$$\left(1 + \frac{9}{16}\right)x^2 + \left(-\frac{51}{4} - 6\right)x + \left(\frac{289}{4} - 16\right) = 0$$
$$\frac{25}{16}x^2 - \frac{75x}{4} + \frac{225}{4} = 0$$
$$x = 6 \text{ (repeated)} \qquad \text{1M}$$

Thus, L is a tangent to C. 1

(b) By (a), when $x = 6$, $y = -\dfrac{3(6)}{4} + \dfrac{17}{2} = 4$.

The coordinates of A are $(6,\ 4)$. 1A

(c) Let G be the centre of circle C. Coordinates of G are $(3,\ 0)$. 1M

Let $B\ (r,\ s)$ be the point of contact of L_1 and C.

Then AGB is a diameter and so G is midpoint of AB.

$$\frac{6+r}{2} = 3 \qquad \frac{4+s}{2} = 0 \qquad \text{1M}$$
$$r = 0 \qquad s = -4$$

The coordinates of the point of contact are $(0,\ -4)$. 1A

16. (a) Let slope of a straight line through R be m. It has an equation $y = mx - 3$.

$$x^2 + (mx - 3)^2 + 8x + 6(mx - 3) + 17 = 0$$
$$(1 + m^2)x^2 + (-6m + 8 + 6m)x + (9 - 18 + 17) = 0$$
$$(1 + m^2)x^2 + 8x + 8 = 0$$

Since $y = mx - 3$ is a tangent to the circle,

$$8^2 - 4(1 + m^2)(8) = 0$$ 1M
$$m^2 = 1$$
$$m = \pm 1$$

The equation of L_1 and L_2 are $y = -x - 3$ and $y = x - 3$ respectively. 1A+1A

(b) By (a),

$$(1 + 1)x^2 + 8x + 8 = 0$$
$$x^2 + 4x + 4 = 0$$
$$x = -2$$

Put $x = -2$ into L_1, $y = -1$. The coordinates of P are $(-2, -1)$. 1A

Put $x = -2$ into L_2, $y = -5$. The coordinates of Q are $(-2, -5)$. 1A

(c) Area of $\triangle PQR = \frac{1}{2}(-1 + 5)(2)$ 1M

$= 4$ 1A

17. (a) (i) Since G is centroid of $\triangle OAB$, P and Q are midpoints of AB and OA respectively.

$PQ // OB$ and $QP = \frac{1}{2}OB$ *(mid-point theorem)*

In $\triangle PGQ$ and $\triangle OGB$,

$\angle PGQ = \angle OGB$ *(vert. opp. ∠s)*

$\angle PQG = \angle OBG$ *(alt. ∠s, QP//OB)*

$\angle QPG = \angle BOG$ *(alt. ∠s, QP//OB)*

$\triangle PGQ \sim \triangle OGB$ *(AAA)*

Marking Scheme		
Case 1	Any correct proof with correct reasons.	3
Case 2	Any correct proof without reasons.	2
Case 3	Incomplete proof with any one correct step with reason.	1

(ii) $\triangle PGQ \sim \triangle OGB$ *(proved)*

$$\frac{GQ}{BG} = \frac{QP}{OB} \quad \text{(corr. sides., } \sim \triangle\text{s)}$$

$$= \frac{\frac{1}{2}OB}{OB} \quad \text{(proved)}$$

$$= \frac{1}{2}$$

Thus, $BG : GQ = 2 : 1$.

Marking Scheme		
Case 1	Any correct proof with correct reasons.	2
Case 2	Any correct proof without reasons.	1

(b) (i) Let (m, n) and $(b, 0)$ be the coordinates of Q and B respectively.

Since $BG : GQ = 2 : 1$,

$$\frac{n-1}{1-0} = \frac{1}{2}$$

$$n = \frac{3}{2}$$

Since Q lies on OA, when $n = \frac{3}{2}$, $m = 2 \times \frac{3}{2} = 3$.

The coordinates of Q are $\left(3, \frac{3}{2}\right)$. 1M

$$\frac{b-5}{5-3} = \frac{2}{1}$$

$$b = 9$$

The coordinates of B are $(9, 0)$. 1A

(ii) Let $S(p, q)$ be the circumcentre of $\triangle OAB$.

Since S lies on the perpendicular bisector of OB, $p = \frac{9}{2}$. 1M

Since $OA \perp SQ$,

$$\frac{1}{2} \times \frac{q - \frac{3}{2}}{\frac{9}{2} - 3} = -1 \qquad \text{1M+1M}$$

$$q = -\frac{3}{2}$$

The coordinates of the circumcentre are $\left(\frac{9}{2},\ -\frac{3}{2}\right)$. 1A

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