M2REG-MAD-2223-ASM-SET 1-MATH
Suggested solutions

1. (a) $A + B + C = \mathbf{0}$

$$C = -A - B \qquad \text{1M}$$

$$= -\begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & 1 \end{pmatrix} - \begin{pmatrix} 7 & -1 & 3 \\ 4 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & -1 & -2 \\ -8 & -7 & 0 \end{pmatrix} \qquad \text{1A}$$

(b) $$A + B - 2C = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & 1 \end{pmatrix} + \begin{pmatrix} 7 & -1 & 3 \\ 4 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 16 & 2 & 4 \\ 16 & 14 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 24 & 3 & 6 \\ 24 & 21 & 0 \end{pmatrix} \qquad \text{1A}$$

2. $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 5 & -c \\ a & 2a \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -3 & 7 \end{pmatrix}$$

$$\begin{pmatrix} a+5 & b-c \\ c+a & d+2a \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -3 & 7 \end{pmatrix}$$

We have $a + 5 = 6$, $b - c = -4$, $c + a = -3$ and $d + 2a = 7$. 1M

Solving, we have $a = 1$, $b = -8$, $c = -4$ and $d = 5$. 1A+1A

3. (a) $$A + B = \begin{pmatrix} 5 & 2 \\ -1 & -4 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 1 & 7 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 0 & 3 \\ -1 & 3 \end{pmatrix} \qquad \text{1A}$$

(b) $$B - C = \begin{pmatrix} 4 & 0 \\ 1 & 7 \\ -3 & 2 \end{pmatrix} - \begin{pmatrix} 6 & 3 \\ -1 & 1 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 2 & 6 \\ 2 & 1 \end{pmatrix} \qquad \text{1A}$$

(c) $$2C - (B - A) = 2\begin{pmatrix} 6 & 3 \\ -1 & 1 \\ -5 & 1 \end{pmatrix} - \left[\begin{pmatrix} 4 & 0 \\ 1 & 7 \\ -3 & 2 \end{pmatrix} - \begin{pmatrix} 5 & 2 \\ -1 & -4 \\ 2 & 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 13 & 8 \\ -4 & -9 \\ -5 & 1 \end{pmatrix} \qquad \text{1A}$$

4. $$-\frac{1}{3}\begin{pmatrix} 3 & -9 \\ 12 & 0 \end{pmatrix} - 4\begin{pmatrix} -1 & -1 \\ 0 & 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 6 & 4 \\ -8 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -4 & 0 \end{pmatrix} - \begin{pmatrix} -4 & -4 \\ 0 & 12 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ -4 & -1 \end{pmatrix} \qquad \text{1A}$$

$$= \begin{pmatrix} 6 & 9 \\ -8 & -13 \end{pmatrix} \qquad \text{1A}$$

5. (a) $(2A - B) - 2(A + 3B) = \begin{pmatrix} 19 & 4 \\ 1 & 0 \end{pmatrix} - 2\begin{pmatrix} -1 & 2 \\ 32 & -7 \end{pmatrix}$ 1M

$$-7B = \begin{pmatrix} 21 & 0 \\ -63 & 14 \end{pmatrix}$$

$$B = \begin{pmatrix} -3 & 0 \\ 9 & -2 \end{pmatrix}$$ 1A

$$A + 3B = \begin{pmatrix} -1 & 2 \\ 32 & -7 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 2 \\ 32 & -7 \end{pmatrix} - 3B$$

$$= \begin{pmatrix} -1 & 2 \\ 32 & -7 \end{pmatrix} - 3\begin{pmatrix} -3 & 0 \\ 9 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 2 \\ 5 & -1 \end{pmatrix}$$ 1A

(b) $2A = \begin{pmatrix} 16 & 4 \\ 10 & -2 \end{pmatrix}$

$$B^2 = \begin{pmatrix} -3 & 0 \\ 9 & -2 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 9 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 0 \\ -45 & 4 \end{pmatrix}$$ 1A

$$2A + B^2 = \begin{pmatrix} 16 & 4 \\ 10 & -2 \end{pmatrix} + \begin{pmatrix} 9 & 0 \\ -45 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 4 \\ -35 & 2 \end{pmatrix}$$ 1A

6. $A^2 = \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix} = \begin{pmatrix} 16 & -9 \\ -12 & 37 \end{pmatrix}$ 1A

$$A^2 + rA = sI$$

$$\begin{pmatrix} 16 & -9 \\ -12 & 37 \end{pmatrix} + \begin{pmatrix} 2r & 3r \\ 4r & -5r \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

$$\begin{pmatrix} 16 + 2r & 3r - 9 \\ 4r - 12 & 37 - 5r \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$ 1A

We have $16 + 2r = s$ and $3r - 9 = 0$. 1M

Solving, we have $r = 3$ and $s = 22$. 1A

7. (a) $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix}$ 1A

Thus, $\alpha = 1$. 1A

$\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \beta\begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$\begin{pmatrix} -8 \\ 8 \end{pmatrix} = \begin{pmatrix} -2\beta \\ 2\beta \end{pmatrix}$

Thus, $\beta = 4$. 1A

(b) L.H.S. $= \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -8 \\ 2 & 8 \end{pmatrix}$ 1A

R.H.S. $= \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -8 \\ 2 & 8 \end{pmatrix}$

Thus, $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. 1

8. $A^2 = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 2 & 2 \\ -1 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & -3 & 3 \\ 0 & 2 & 2 \\ -1 & 5 & 4 \end{pmatrix}$

$= \begin{pmatrix} -2 & 6 & 9 \\ -2 & 14 & 12 \\ -5 & 33 & 23 \end{pmatrix}$ 1A

$A^3 = \begin{pmatrix} -2 & 6 & 9 \\ -2 & 14 & 12 \\ -5 & 33 & 23 \end{pmatrix}\begin{pmatrix} 1 & -3 & 3 \\ 0 & 2 & 2 \\ -1 & 5 & 4 \end{pmatrix}$

$= \begin{pmatrix} -11 & 63 & 42 \\ -14 & 94 & 70 \\ -28 & 196 & 143 \end{pmatrix}$ 1A

9. (a) $AB = \begin{pmatrix} 1 & 0 & 2 \\ -2 & -1 & 2 \\ 4 & 2 & 0 \end{pmatrix}\begin{pmatrix} 4 & -4 & -2 \\ -8 & 8 & 6 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ 1A

$BA = \begin{pmatrix} 4 & -4 & -2 \\ -8 & 8 & 6 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ -2 & -1 & 2 \\ 4 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Thus, $AB = BA = 4I$. 1

(b) (i) $A(BC) + (CB)A = (AB)C + C(BA)$

$= (4I)C + C(4I)$

$= 8C$ 1M

$= \begin{pmatrix} -8 & 16 & 8 \\ 16 & -64 & 0 \\ -16 & 24 & 48 \end{pmatrix}$ 1A

(ii) $ABCBA = (AB)C(BA)$

$= (4I)C(4I)$

$= 16C$ 1M

$= \begin{pmatrix} -16 & 32 & 16 \\ 32 & -128 & 0 \\ -32 & 48 & 96 \end{pmatrix}$ 1A

10. (a) $A^2 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 2a \\ 0 & a^2 \end{pmatrix}$ 1A

$A^3 = \begin{pmatrix} a^2 & 2a \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^3 & 3a^2 \\ 0 & a^3 \end{pmatrix}$ 1A

$A^4 = \begin{pmatrix} a^3 & 3a^2 \\ 0 & a^3 \end{pmatrix}\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^4 & 4a^3 \\ 0 & a^4 \end{pmatrix}$ 1A

(b) Proposition: $A^n = \begin{pmatrix} a^n & na^{n-1} \\ 0 & a^n \end{pmatrix}$.

When $n = 1$,

R.H.S. $= \begin{pmatrix} a & (1)a^{1-1} \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ = L.H.S.

The statement is true for $n = 1$. 1

Assume $A^k = \begin{pmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{pmatrix}$, where $k \in \mathbb{Z}^+$. 1

$A^{k+1} = A^k A$

$= \begin{pmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{pmatrix}\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ 1

$= \begin{pmatrix} a^{k+1} & (k+1)a^k \\ 0 & a^{k+1} \end{pmatrix}$ 1

The statement is true for $n = k + 1$.

By M.I., the statement is true $\forall n \in \mathbb{Z}^+$. 1

(c) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^{10} = \begin{pmatrix} 2^{10} & 10 \cdot 2^{10-1} \\ 0 & 2^{10} \end{pmatrix}$ 1M

$= \begin{pmatrix} 1024 & 5120 \\ 0 & 1024 \end{pmatrix}$ 1A

11. (a) $A^2 - 5A + 4I = \begin{pmatrix} 6 & 10 \\ 5 & 11 \end{pmatrix} - 5\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ 1M

$= \mathbf{0}$ 1A

(b) When $n = 1$, R.H.S. $= \dfrac{4-1}{3}A + \dfrac{4-4}{3}I = A =$ L.H.S.

The statement is true for $n = 1$. 1

Assume $A^k = \left(\dfrac{4^k - 1}{3}\right)A + \left(\dfrac{4 - 4^k}{3}\right)I$, where $k \in \mathbb{Z}^+$. 1

$$\begin{aligned} A^{k+1} &= A^k A \\ &= \left[\left(\frac{4^k - 1}{3}\right)A + \left(\frac{4 - 4^k}{3}\right)I\right]A \qquad 1 \\ &= \left(\frac{4^k - 1}{3}\right)A^2 + \left(\frac{4 - 4^k}{3}\right)A \\ &= \left(\frac{4^k - 1}{3}\right)(5A - 4I) + \left(\frac{4 - 4^k}{3}\right)A \\ &= \left(\frac{(5-1)4^k + (-5+4)}{3}\right)A + \left(\frac{4 - 4^{k+1}}{3}\right)I \\ &= \left(\frac{4^{k+1} - 1}{3}\right)A + \left(\frac{4 - 4^{k+1}}{3}\right)I \qquad 1 \end{aligned}$$

The statement is true for $n = k + 1$.

By M.I., the statement is true $\forall n \in \mathbb{Z}^+$. 1

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