Dexter Wong & His Mathematics Team

暑期課程　2022 – 2023

數學　延伸部分

S5 – S6 M2 積分　功課　第 8 套

名字：______________　　分校：__________

上課日：日 / 一 / 二 / 三 / 四 / 五 / 六

考生須知

1. 本試卷**各題均須作答**，答案須寫在本試題答題簿中預留的空位內。
2. 除特別指明外，須詳細列出所有算式。
3. 除特別指明外，數值答案須用真確值表示。
4. 本試卷的附圖不一定依比例繪成。

建議題解

派發於暑期課程
S5 – S6 M2 積分
第 2 期 – 第 4 堂

1. (a) 設 $u = a + b - x$。則 $\mathrm{d}u = -\mathrm{d}x$。 1M

$$\int_a^b f(a+b-x)\,\mathrm{d}x = -\int_b^a f(u)\,\mathrm{d}u$$

$$= \int_a^b f(u)\,\mathrm{d}u \quad \text{1M}$$

$$= \int_a^b f(x)\,\mathrm{d}x \quad \text{1}$$

(b) (i) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln(\tan x)\,\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln\left[\tan\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right]\mathrm{d}x \quad \text{1M}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln\left(\frac{1}{\tan x}\right)\mathrm{d}x$$

$$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln(\tan x)\,\mathrm{d}x \quad \text{1A}$$

$$2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln(\tan x)\,\mathrm{d}x = 0$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln(\tan x)\,\mathrm{d}x = 0 \quad \text{1A}$$

(ii) $$\int_5^{10} \frac{x^{10}}{x^{10}+(15-x)^{10}}\,\mathrm{d}x = \int_5^{10} \frac{(5+10-x)^{10}}{(5+10-x)^{10}+[15-(5+10-x)]^{10}}\,\mathrm{d}x \quad \text{1M}$$

$$= \int_5^{10} \frac{(15-x)^{10}}{(15-x)^{10}+x^{10}}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_5^{10} \frac{(15-x)^{10}}{(15-x)^{10}+x^{10}}\,\mathrm{d}x + \int_5^{10} \frac{x^{10}}{x^{10}+(15-x)^{10}}\,\mathrm{d}x\right]$$

$$= \frac{1}{2}\int_5^{10} \mathrm{d}x \quad \text{1A}$$

$$= \frac{1}{2}\Big[x\Big]_5^{10}$$

$$= \frac{5}{2} \quad \text{1A}$$

2. (a) (i)

$$\begin{aligned}
\int \sin^4 x\,dx &= \int \sin^3 x \sin x\,dx \\
&= -\sin^3 x\cos x + 3\int \sin^2 x\cos^2 x\,dx && \text{1M} \\
&= -\sin^3 x\cos x + 3\int \sin^2 x(1-\sin^2 x)\,dx \\
&= -\sin^3 x\cos x + 3\int \sin^2 x\,dx - 3\int \sin^4 x\,dx \\
4\int \sin^4 x\,dx &= -\sin^3 x\cos x + 3\int \sin^2 x\,dx \\
\int \sin^4 x\,dx &= -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\int \sin^2 x\,dx && \text{1}
\end{aligned}$$

(ii)

$$\begin{aligned}
\int_0^{2\pi} \sin^4 x\,dx &= -\frac{1}{4}\Big[\cos x\sin^3 x\Big]_0^{2\pi} + \frac{3}{4}\int_0^{2\pi} \sin^2 x\,dx \\
&= \frac{3}{8}\int_0^{2\pi}(1-\cos 2x)\,dx && \text{1M} \\
&= \frac{3}{8}\left[x - \frac{\sin 2x}{2}\right]_0^{2\pi} && \text{1M} \\
&= \frac{3\pi}{4} && \text{1A}
\end{aligned}$$

(b) (i) 設 $u = a - x$。則 $du = -dx$。 1M

$$\begin{aligned}
\int_0^a xf(x)\,dx &= -\int_a^0 (a-u)f(u)\,du && \text{1M} \\
&= \int_0^a (a-u)f(u)\,du \\
&= a\int_0^x f(x)\,dx - \int_0^a xf(x)\,dx \\
2\int_0^a xf(x)\,dx &= a\int_0^a f(x)\,dx \\
\int_0^a xf(x)\,dx &= \frac{a}{2}\int_0^a f(x)\,dx && \text{1}
\end{aligned}$$

(ii) 代 $f(x) = \sin^4 x$。則 $f(2\pi - x) = \sin^4(2\pi - x) = \sin^4 x = f(x)$。 1M

$$\begin{aligned}
\int_0^{2\pi} x\sin^4 x\,dx &= \frac{2\pi}{2}\int_0^{2\pi} \sin^4 x\,dx && \text{1M} \\
&= \pi\left(\frac{3\pi}{4}\right) \\
&= \frac{3\pi^2}{4} && \text{1A}
\end{aligned}$$

3. (a) 設 $u=\frac{\pi}{4}-x$。則 $du=-dx$。 1M

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\ln\left[\cos\left(\frac{\pi}{4}-x\right)\right]dx=-\int_{\frac{\pi}{6}}^{\frac{\pi}{12}}\ln(\cos u)\,du$$

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\ln(\cos u)\,du \qquad \text{1M}$$

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\ln(\cos x)\,dx \qquad \text{1}$$

(b)

$$0=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\ln\left[\cos\left(\frac{\pi}{4}-x\right)\right]dx-\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\ln(\cos x)\,dx$$

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\ln\left[\frac{\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x}{\cos x}\right]dx \qquad \text{1M}$$

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\ln\left[\frac{1}{\sqrt{2}}(1+\tan x)\right]dx$$

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\left[\ln(1+\tan x)-\frac{1}{2}\ln 2\right]dx \qquad \text{1M}$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\ln(1+\tan x)\,dx=\frac{1}{2}\ln 2\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}dx$$

$$=\frac{1}{2}\ln 2\Big[x\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$$

$$=\frac{\pi\ln 2}{24} \qquad \text{1A}$$

(c) (i)

$$\tan\frac{\pi}{12}=\tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$=\frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}} \qquad \text{1M}$$

$$=\frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}}$$

$$=\frac{3-\sqrt{3}}{3+\sqrt{3}}\times\frac{3-\sqrt{3}}{3-\sqrt{3}}$$

$$=\frac{9-2\sqrt{3}+3}{9-3}$$

$$=2-\sqrt{3} \qquad \text{1}$$

(ii) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x \sec^2 x}{1 + \tan x}\, dx$

$$= \Big[x \ln(1 + \tan x) \Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}} - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(1 + \tan x)\, dx \qquad \text{1M}$$

$$= \frac{\pi}{6} \ln\left(1 + \tan\frac{\pi}{6}\right) - \frac{\pi}{12} \ln\left(1 + \tan\frac{\pi}{12}\right) - \frac{\pi \ln 2}{24} \qquad \text{1M}$$

$$= \frac{\pi}{6} \ln\left(1 + \frac{1}{\sqrt{3}}\right) - \frac{\pi}{12} \ln\left(3 - \sqrt{3}\right) - \frac{\pi \ln 2}{24} \qquad \text{1M}$$

$$= \frac{\pi}{12} \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)^2 - \ln\left(3 - \sqrt{3}\right)\Big] - \frac{\pi \ln 2}{24}$$

$$= \frac{\pi}{12} \ln \frac{(\sqrt{3}+1)^2}{3(3-\sqrt{3})} - \frac{\pi \ln 2}{24} \qquad \text{1M}$$

$$= \frac{\pi}{12} \ln\left[\frac{4 + 2\sqrt{3}}{3(3-\sqrt{3})} \cdot \frac{3+\sqrt{3}}{3+\sqrt{3}}\right] - \frac{\pi \ln 2}{24}$$

$$= \frac{\pi}{12} \ln \frac{9 + 5\sqrt{3}}{9} - \frac{\pi \ln 2}{24}$$

$$= \frac{\pi}{24} \ln\left[\left(\frac{9 + 5\sqrt{3}}{9}\right)^2 \div 2\right]$$

$$= \frac{\pi}{24} \ln \frac{81 + 90\sqrt{3} + 75}{162}$$

$$= \frac{\pi}{24} \ln \frac{26 + 15\sqrt{3}}{27} \qquad \text{1}$$

4. (a) 設 $x=\sqrt{3}\tan\theta$。則 $\mathrm{d}x=\sqrt{3}\sec^2\theta\,\mathrm{d}\theta$。 1M

$$\int_0^1 \frac{1}{x^2+3}\,\mathrm{d}x = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2\theta}{3\tan^2\theta+3}\,\mathrm{d}\theta$$

$$= \frac{\sqrt{3}}{3}\int_0^{\frac{\pi}{6}} \mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\sqrt{3}}{3}\Big[\theta\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}\pi}{18} \qquad \text{1A}$$

(b) (i)

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$$

$$= \frac{\cos 2\theta}{1}$$

$$= \cos 2\theta \qquad \text{1}$$

(ii) 設 $t=\tan\theta$。則 $\mathrm{d}t=\sec^2\theta\,\mathrm{d}\theta$ 及 $\mathrm{d}\theta=\dfrac{1}{1+t^2}\,\mathrm{d}t$。 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta+2}\,\mathrm{d}\theta = \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2}+2}\cdot\frac{1}{1+t^2}\,\mathrm{d}t \qquad \text{1M}$$

$$= \int_0^1 \frac{1}{t^2+3}\,\mathrm{d}t$$

$$= \frac{\sqrt{3}\pi}{18} \qquad \text{1A}$$

(c) 設 $u=\dfrac{\pi}{4}-\theta$。則 $\mathrm{d}u=-\mathrm{d}\theta$。 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta+2}\,\mathrm{d}\theta = -\int_{\frac{\pi}{4}}^0 \frac{1}{\cos\left[\frac{\pi}{2}-2u\right]+2}\,\mathrm{d}u$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2u+2}\,\mathrm{d}u$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+2}\,\mathrm{d}\theta \qquad \text{1}$$

(d) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + \cos 2\theta + 4}{(\sin 2\theta + 2)(\cos 2\theta + 2)}\, d\theta$

$\displaystyle = \int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 2}{(\sin 2\theta + 2)(\cos 2\theta + 2)}\, d\theta + \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 2}{(\sin 2\theta + 2)(\cos 2\theta + 2)}\, d\theta$ 1M

$\displaystyle = \int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 2}\, d\theta + \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + 2}\, d\theta$

$\displaystyle = 2\int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 2}\, d\theta$

$\displaystyle = 2 \times \frac{\sqrt{3}\pi}{18}$ 1M

$\displaystyle = \frac{\sqrt{3}\pi}{9}$ 1A

5. (a) 設 $u = a - x$。則 $du = -dx$。 1M

$\displaystyle \int_0^{a} f(a-x)\, dx = -\int_a^{0} f(u)\, du$

$\displaystyle = \int_0^{a} f(u)\, du$ 1M

$\displaystyle = \int_0^{a} f(x)\, dx$ 1

(b) (i) $\displaystyle \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \cos\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right) + \cos\left(\frac{\pi}{2} - \theta\right)}\, d\theta$

$\displaystyle = \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sin\theta + \cos\theta}\, d\theta$ 1

(ii) $\displaystyle \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \cos\theta}\, d\theta = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sin\theta + \cos\theta}\, d\theta + \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \cos\theta}\, d\theta\right]$ 1M

$\displaystyle = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta$

$\displaystyle = \frac{1}{2}\Big[\theta\Big]_0^{\frac{\pi}{2}}$

$\displaystyle = \frac{\pi}{4}$ 1A

(c) 設 $x = \sin\theta$。則 $dx = \cos\theta\, d\theta$。 1M

$\displaystyle \int_0^{1} \frac{dx}{x + \sqrt{1 - x^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \sqrt{1 - \sin^2\theta}}\, d\theta$

$\displaystyle = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \cos\theta}\, d\theta$

$\displaystyle = \frac{\pi}{4}$ 1A