Dexter Wong & His Mathematics Team

暑期課程　2022 – 2023

數學　延伸部分

S5 – S6 M2 積分　功課　第 7 套

名字：＿＿＿＿＿＿＿＿＿＿＿＿　　分校：＿＿＿＿＿＿

上課日：日 / 一 / 二 / 三 / 四 / 五 / 六

考生須知

1. 本試卷**各題均須作答**，答案須寫在本試題答題簿中預留的空位內。
2. 除特別指明外，須詳細列出所有算式。
3. 除特別指明外，數值答案須用真確值表示。
4. 本試卷的附圖不一定依比例繪成。

建議題解

派發於暑期課程
S5 – S6 M2 積分
第 2 期 – 第 3 堂

1. (a) (i) $\cos\theta(2\cos 4\theta - 2\cos 2\theta + 1) = 2\cos\theta\cos 4\theta - 2\cos\theta\cos 2\theta + \cos\theta$

$= (\cos 5\theta + \cos 3\theta) - (\cos 3\theta + \cos\theta) + \cos\theta$ 1M

$= \cos 5\theta$

因此，$\dfrac{\cos 5\theta}{\cos\theta} = 2\cos 4\theta - 2\cos 2\theta + 1$。 1

(ii) $\dfrac{\cos 5\left(\frac{\pi}{4} - t\right)}{\cos\left(\frac{\pi}{4} - t\right)} = 2\cos\left(\frac{\pi}{4} - t\right) - 2\cos 2\left(\frac{\pi}{4} - t\right) + 1$

$\dfrac{\cos\frac{5\pi}{4}\cos 5t + \sin\frac{5\pi}{4}\sin 5t}{\cos\frac{\pi}{4}\cos t + \sin\frac{\pi}{4}\sin t} = -2\cos 4t - 2\sin 2t + 1$ 1M

$\dfrac{-\frac{\sqrt{2}}{2}\cos 5t - \frac{\sqrt{2}}{2}\sin 5t}{\frac{\sqrt{2}}{2}\cos t + \frac{\sqrt{2}}{2}\sin t} = -2\cos 4t - 2\sin 2t + 1$

$\dfrac{\cos 5t + \sin 5t}{\cos t + \sin t} = 2\cos 4t + 2\sin 2t - 1$ 1

(b) (i) 設 $u = a - x$。則 $\mathrm{d}u = -\mathrm{d}x$。 1M

$\displaystyle\int_0^a f(a-x)\,\mathrm{d}x = -\int_a^0 f(u)\,\mathrm{d}u$

$\displaystyle= \int_0^a f(u)\,\mathrm{d}u$ 1M

$\displaystyle= \int_0^a f(x)\,\mathrm{d}x$ 1

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos 5x}{\cos x + \sin x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos 5\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)}$ 1M

$\displaystyle= \int_0^{\frac{\pi}{2}} \frac{\sin 5x}{\cos x + \sin x}\,\mathrm{d}x$ 1

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos 5x}{\cos x + \sin x}\,\mathrm{d}x$

$\displaystyle= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\cos 5x}{\cos x + \sin x}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{\sin 5x}{\cos x + \sin x}\,\mathrm{d}x\right]$ 1M

$\displaystyle= \frac{1}{2}\int_0^{\frac{\pi}{2}} (2\cos 4x + 2\sin 2x - 1)\,\mathrm{d}x$ 1M

$\displaystyle= \frac{1}{2}\left[\frac{\sin 4x}{2} - \cos 2x - x\right]_0^{\frac{\pi}{2}}$

$\displaystyle= 1 - \frac{\pi}{4}$ 1A

2. (a) 設 $u = -x$。則 $\mathrm{d}u = -\mathrm{d}x$。 1M

$$\int_{-a}^{0} f(x)\,\mathrm{d}x = -\int_{a}^{0} f(-u)\,\mathrm{d}u$$

$$= \int_{0}^{a} f(-u)\,\mathrm{d}u \quad \text{1M}$$

$$= \int_{0}^{a} f(-x)\,\mathrm{d}x \quad \text{1}$$

(b) $$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x$$

$$= \int_{0}^{a} f(-x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x \quad \text{1M}$$

$$= \int_{0}^{a} [f(x) + f(-x)]\,\mathrm{d}x \quad \text{1}$$

(c) 代 $f(x) = \ln\left(\sqrt{e^6 + x^6} - x^3\right) + \cos\dfrac{\pi x}{2}$。

$f(x) + f(-x)$

$$= \left[\ln\left(\sqrt{e^6 + x^6} - x^3\right) + \cos\frac{\pi x}{2}\right] + \left[\ln\left(\sqrt{e^6 + (-x)^6} - (-x)^3\right) + \cos\frac{\pi(-x)}{2}\right]$$

$$= \ln[(\sqrt{e^6 + x^6} - x^3)(\sqrt{e^6 + x^6} + x^3)] + 2\cos\frac{\pi x}{2}$$

$$= \ln[(e^6 + x^6) - x^6] + 2\cos\frac{\pi x}{2}$$

$$= 6 + 2\cos\frac{\pi x}{2} \quad \text{1}$$

$$\int_{-3}^{3} \left[\ln\left(\sqrt{e^6 + x^6} - x^3\right) + \cos\frac{\pi x}{2}\right]\mathrm{d}x$$

$$= \int_{0}^{3} [f(x) + f(-x)]\,\mathrm{d}x \quad \text{1M}$$

$$= \int_{0}^{3} \left(6 + 2\cos\frac{\pi x}{2}\right)\mathrm{d}x$$

$$= \left[6x + \frac{4}{\pi}\sin\frac{\pi x}{2}\right]_0^3$$

$$= 18 - \frac{4}{\pi} \quad \text{1A}$$

3. (a) 設 $u = -x$。則 $\mathrm{d}u = -\mathrm{d}x$。 1M

$$\begin{aligned}\int_{-a}^{a} f(x)\,\mathrm{d}x &= \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x && \text{1M}\\ &= -\int_{a}^{0} f(-u)\,\mathrm{d}u + \int_{0}^{a} f(x)\,\mathrm{d}x\\ &= \int_{0}^{a} f(-u)\,\mathrm{d}u + \int_{0}^{a} f(x)\,\mathrm{d}x && \text{1M}\\ &= \int_{0}^{a} f(-x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x\\ &= \int_{0}^{a} [f(x) + f(-x)]\,\mathrm{d}x && \text{1}\end{aligned}$$

(b) $\displaystyle\int_{-\pi}^{\pi} \ln\left(\sqrt{e^3 + \sin^2 x} - \sin x\right)\mathrm{d}x$

$$= \int_{0}^{\pi} \left[\ln\left(\sqrt{e^3 + \sin^2 x} - \sin x\right) + \ln\left(\sqrt{e^3 + (-\sin x)^2} - (-\sin x)\right)\right]\mathrm{d}x \quad \text{1M}$$

$$= \int_{0}^{\pi} \left[\ln\left(\sqrt{e^3 + \sin^2 x} - \sin x\right) + \ln\left(\sqrt{e^3 + \sin^2 x} + \sin x\right)\right]\mathrm{d}x \quad \text{1}$$

(c) $\displaystyle\int_{-\pi}^{\pi} \ln\left(\sqrt{e^3 + \sin^2 x} - \sin x\right)\mathrm{d}x$

$$= \int_{0}^{\pi} \left[\ln\left(\sqrt{e^3 + \sin^2 x} - \sin x\right) + \ln\left(\sqrt{e^3 + \sin^2 x} + \sin x\right)\right]\mathrm{d}x$$

$$= \int_{0}^{\pi} \ln\left[\left(\sqrt{e^3 + \sin^2 x} - \sin x\right)\left(\sqrt{e^3 + \sin^2 x} + \sin x\right)\right]\mathrm{d}x \quad \text{1M}$$

$$= \int_{0}^{\pi} \ln\left[\left(e^3 + \sin^2 x\right) - \sin^2 x\right]\mathrm{d}x$$

$$= \int_{0}^{\pi} \ln e^3\,\mathrm{d}x$$

$$= 3\int_{0}^{\pi} \mathrm{d}x \quad \text{1A}$$

$$= 3\Big[x\Big]_{0}^{\pi}$$

$$= 3\pi \quad \text{1A}$$

(d) 設 $u = x - \pi$。則 $\mathrm{d}x = \mathrm{d}x$。 1M

$$\int_0^{2\pi} \ln\left(e^3 + 2\sin^2 x + 2\sin x\sqrt{e^3 + \sin^2 x}\right)\mathrm{d}x$$

$$= \int_{-\pi}^{\pi} \ln\left(e^3 + 2\sin^2(\pi + u) + 2\sin(\pi + u)\sqrt{e^3 + \sin^2(\pi + u)}\right)\mathrm{d}u$$

$$= \int_{-\pi}^{\pi} \ln\left((e^3 + \sin^2 u) - 2\sin u\sqrt{e^3 + \sin^2 u} + \sin^2 u\right)\mathrm{d}u \quad \text{1M}$$

$$= \int_{-\pi}^{\pi} \ln\left(\sqrt{e^3 + \sin^2 u} - \sin u\right)^2 \mathrm{d}x$$

$$= 2\int_{-\pi}^{\pi} \ln\left(\sqrt{e^3 + \sin^2 u} - \sin u\right)\mathrm{d}u \quad \text{1M}$$

$$= 2(3\pi)$$

$$= 6\pi \quad \text{1A}$$

4. (a) 設 $u = a - x$。則 $\mathrm{d}u = -\mathrm{d}x$。 1M

$$\begin{aligned}
\frac{1}{2}\int_0^a [f(x) + f(a-x)]\,\mathrm{d}x &= \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{2}\int_0^a f(a-x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x - \frac{1}{2}\int_a^0 f(u)\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{2}\int_0^a f(u)\,\mathrm{d}u \qquad \text{1M} \\
&= \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x \\
&= \int_0^a f(x)\,\mathrm{d}x \qquad \text{1}
\end{aligned}$$

(b) $x^2 - 6x + 18 = (x-3)^2 + 9$

設 $x - 3 = 3\tan\theta$。則 $\mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta$。 1M

$$\begin{aligned}
\int_0^6 \frac{\mathrm{d}x}{x^2 - 6x + 18} &= \int_0^6 \frac{\mathrm{d}x}{(x-3)^2 + 9} \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{3\sec^2\theta}{9\tan^2\theta + 9}\,\mathrm{d}\theta \qquad \text{1M} \\
&= \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{d}\theta \qquad \text{1A} \\
&= \frac{1}{3}\Big[\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{\pi}{6} \qquad \text{1A}
\end{aligned}$$

(c) Put $f(x) = \dfrac{1}{(x^2 - 6x + 18)(e^{3-x} + 1)}$。

$$\begin{aligned}
&f(x) + f(6-x) \\
&= \frac{1}{(x^2 - 6x + 18)(e^{3-x} + 1)} + \frac{1}{[(6-x)^2 - 6(6-x) + 18](e^{3-(6-x)} + 1)} \\
&= \frac{1}{(x^2 - 6x + 18)(e^{3-x} + 1)} + \frac{1}{(x^2 - 6x + 18)(e^{-(3-x)} + 1)} \\
&= \frac{1}{(x^2 - 6x + 18)(e^{3-x} + 1)} + \frac{e^{3-x}}{(x^2 - 6x + 18)(1 + e^{3-x})} \\
&= \frac{1}{x^2 - 6x + 18} \qquad \text{1A}
\end{aligned}$$

$$\begin{aligned}
&\int_0^6 \frac{\mathrm{d}x}{(x^2 - 6x + 18)(e^{3-x} + 1)} \\
&= \frac{1}{2}\int_0^6 [f(x) + f(6-x)]\,\mathrm{d}x \qquad \text{1M} \\
&= \frac{1}{2}\int_0^6 \frac{\mathrm{d}x}{x^2 - 6x + 18} \\
&= \frac{1}{2}\left(\frac{\pi}{6}\right) \qquad \text{1M} \\
&= \frac{\pi}{12} \qquad \text{1A}
\end{aligned}$$

5. (a) 設 $u = x^3$。則 $\mathrm{d}u = 3x^2\,\mathrm{d}x$。 1M

$$\begin{aligned}\int x^2 \sin x^3\,\mathrm{d}x &= \frac{1}{3}\int \sin u\,\mathrm{d}u \\ &= -\frac{\cos u}{3} + \text{常數} \\ &= -\frac{\cos x^3}{3} + \text{常數}\end{aligned}$$ 1A

(i) 設 $u = a + b - x$。則 $\mathrm{d}u = -\mathrm{d}x$。 1M

$$\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,\mathrm{d}x$$

$$= -\int_b^a \frac{f(a+b-u)}{f(u)+f(a+b-u)}\,\mathrm{d}u$$

$$= \int_a^b \frac{f(a+b-u)}{f(u)+f(a+b-u)}\,\mathrm{d}u$$ 1M

$$= \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}\,\mathrm{d}x$$ 1A

$$\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,\mathrm{d}x + \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,\mathrm{d}x\right]$$ 1M

$$= \frac{1}{2}\int_a^b \mathrm{d}x$$ 1A

$$= \frac{1}{2}\Big[x\Big]_a^b$$

$$= \frac{b-a}{2}$$ 1

(ii) 設 $u = x^3$。則 $\mathrm{d}u = 3x^2\,\mathrm{d}x$。 1M

$$\int_{\sqrt[3]{\log 5}}^{\sqrt[3]{\log 6}} \frac{x^2 \sin x^3}{\sin(\log 30 - x^3) + \sin x^3}\,\mathrm{d}x$$

$$= \frac{1}{3}\int_{\log 5}^{\log 6} \frac{\sin u}{\sin(\log 5 + \log 6 - u) + \sin u}\,\mathrm{d}u$$ 1M

$$= \frac{1}{3}\cdot\frac{\log 6 - \log 5}{2}$$ 1M

$$= \frac{1}{6}\log\frac{6}{5}$$ 1A

6. (a) 設 $u = a + b - x$。則 $\mathrm{d}u = -\mathrm{d}x$。

$$\int_a^b f(a+b-x)\,\mathrm{d}x = -\int_b^a f(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_a^b f(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_a^b f(x)\,\mathrm{d}x \qquad \text{1}$$

(b) $$\int_0^{\frac{\pi}{4}} \ln\sin 2x\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\sin\left[2\left(\frac{\pi}{4}-x\right)\right]\mathrm{d}x \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{4}} \ln\cos 2x\,\mathrm{d}x \qquad \text{1}$$

(c) 設 $2x = t$。則 $2\,\mathrm{d}x = \mathrm{d}t$。 1M

$$\int_0^{\frac{\pi}{4}} \ln\sin 4x\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin 2t\,\mathrm{d}t$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln\sin 2t\,\mathrm{d}t + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\sin 2t\,\mathrm{d}t\right] \qquad \text{1M}$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln\sin 2t\,\mathrm{d}t + \int_0^{\frac{\pi}{4}} \ln\sin\left[2\left(u+\frac{\pi}{4}\right)\right]\mathrm{d}u\right] \quad \text{where } u = t - \frac{\pi}{4} \qquad \text{1M}$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln\sin 2t\,\mathrm{d}t + \int_0^{\frac{\pi}{4}} \ln\cos 2u\,\mathrm{d}u\right]$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln\sin 2x\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln\sin 2x\,\mathrm{d}x\right]$$

$$= \int_0^{\frac{\pi}{4}} \ln\sin 2x\,\mathrm{d}x \qquad \text{1}$$

(d) $$\int_0^{\frac{\pi}{4}} \ln\sin 4x\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} (\ln 2 + \ln\sin 2x + \ln\cos 2x)\,\mathrm{d}x$$

$$= \ln 2\left[x\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \ln\sin 2x\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln\cos 2x\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi\ln 2}{4} + 2\int_0^{\frac{\pi}{4}} \ln\sin 4x\,\mathrm{d}x \qquad \text{1M}$$

$$\int_0^{\frac{\pi}{4}} \ln\sin 4x\,\mathrm{d}x = -\frac{\pi\ln 2}{4} \qquad \text{1A}$$