

Dexter Wong & His Mathematics Team

暑期課程 2022 – 2023

數學 延伸部分

S5 – S6 M2 積分 功課 第 6 套

名字：_____

分校：_____

上課日：日 / 一 / 二 / 三 / 四 / 五 / 六

考生須知

1. 本試卷各題均須作答，答案須寫在本試題答題簿中預留的空位內。
2. 除特別指明外，須詳細列出所有算式。
3. 除特別指明外，數值答案須用真確值表示。
4. 本試卷的附圖不一定依比例繪成。

建議題解



派發於暑期課程
S5 – S6 M2 積分
第 2 期 – 第 2 堂

$$1. \quad (a) \quad \int \sin(\ln x) \, dx = x \sin(\ln x) - \int x \cos(\ln x) \left(\frac{1}{x}\right) dx \quad 1M$$

$$= x \sin(\ln x) - x \cos(\ln x) + \int x[-\sin(\ln x)] \left(\frac{1}{x}\right) dx \quad 1M$$

$$2 \int \sin(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x) + \text{常數}$$

$$\int \sin(\ln x) \, dx = \frac{x \sin(\ln x)}{2} - \frac{x \cos(\ln x)}{2} + \text{常數} \quad 1A$$

(b) 設 $u = x + 1$ 。則 $du = dx$ 。

$$\begin{aligned} \int_0^{e^\pi - 1} \sin[\ln(x + 1)] \, dx &= \int_1^{e^\pi} \sin(\ln x) \, dx \\ &= \left[\frac{x \sin(\ln x)}{2} - \frac{x \cos(\ln x)}{2} \right]_1^{e^\pi} \quad 1M \end{aligned}$$

$$= \frac{e^\pi + 1}{2} \quad 1A$$

$$2. \quad (a) \quad \frac{x}{x^2 + 4} - \frac{1}{x + 4} = \frac{x(x + 4) - (x^2 + 4)}{(x^2 + 4)(x + 4)} \quad 1M$$

$$= \frac{4x - 4}{(x^2 + 4)(x + 4)}$$

$$= \frac{4(x - 1)}{(x^2 + 4)(x + 4)} \quad 1$$

(b) 設 $t = 2 \tan \theta$ 。則 $dt = 2 \sec^2 \theta \, d\theta$ 。

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{2 \tan \theta - 1}{\tan \theta + 2} \, d\theta &= \int_0^{\frac{\pi}{4}} \frac{2 \tan \theta - 1}{2 \sec^2 \theta (\tan \theta + 2)} \cdot 2 \sec^2 \theta \, d\theta \\ &= \int_0^2 \frac{t - 1}{2 \left(1 + \frac{t^2}{4}\right) \left(\frac{t}{2} + 2\right)} \, dt \quad 1A \end{aligned}$$

$$\begin{aligned} &= \int_0^2 \frac{4(t - 1)}{(t^2 + 4)(t + 4)} \, dt \\ &= \int_0^2 \left(\frac{t}{t^2 + 4} - \frac{1}{t + 4} \right) \, dt \quad 1M \end{aligned}$$

設 $u = t^2 + 4$ 。則 $du = 2t \, dt$ 。 1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{2 \tan \theta - 1}{\tan \theta + 2} \, d\theta &= \frac{1}{2} \int_4^8 \frac{du}{u} - \int_0^2 \frac{dt}{t + 4} \\ &= \frac{1}{2} \left[\ln |u| \right]_4^8 - \left[\ln |t + 4| \right]_0^2 \quad 1A \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} (\ln 8 - \ln 4) - (\ln 6 - \ln 2) \\ &= \frac{5}{2} \ln 2 - \ln 6 \quad 1A \end{aligned}$$

3. (a) 設 $x = 2 \sin \theta$ 。則 $dx = 2 \cos \theta d\theta$ 。 1M

$$\begin{aligned} \int_0^2 \sqrt{4-x^2} dx &= \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2 \theta} (2 \cos \theta) d\theta \\ &= 4 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta & 1M \\ &= 2 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} & 1M \\ &= \pi & 1A \end{aligned}$$

(b) $\int_0^1 x e^x dx = \left[x e^x \right]_0^1 - \int_0^1 e^x dx$ 1M+1A

$$\begin{aligned} &= e - \left[e^x \right]_0^1 \\ &= 1 & 1A \end{aligned}$$

4. (a) 設 $x = 3 \tan \theta$ 。則 $dx = 3 \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int_0^{\sqrt{3}} \frac{1}{x^2+9} dx &= \int_0^{\frac{\pi}{6}} \frac{3 \sec^2 \theta}{9 \tan^2 \theta + 9} d\theta & 1A \\ &= \frac{1}{3} \int_0^{\frac{\pi}{6}} d\theta \\ &= \frac{1}{3} \left[\theta \right]_0^{\frac{\pi}{6}} \\ &= \frac{\pi}{18} & 1A \end{aligned}$$

(b) $\int_0^{\sqrt{3}} \ln(x^2+9) dx = \left[x \ln(x^2+9) \right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{2x^2}{x^2+9} dx$ 1M

$$\begin{aligned} &= \sqrt{3} \ln 12 - 2 \int_0^{\sqrt{3}} \left(1 - \frac{9}{x^2+9} \right) dx & 1M \\ &= \sqrt{3} \ln 12 - 2 \int_0^{\sqrt{3}} dx + 18 \int_0^{\sqrt{3}} \frac{1}{x^2+9} dx \\ &= \sqrt{3} \ln 12 - 2 \left[x \right]_0^{\sqrt{3}} + 18 \left(\frac{\pi}{18} \right) & 1M \\ &= \sqrt{3} \ln 12 - 2\sqrt{3} + \pi & 1A \end{aligned}$$

$$5. \quad (a) \quad \int x \cos kx \, dx = \frac{x \sin kx}{k} - \frac{1}{k} \int \sin kx \, dx \quad 1M$$

$$= \frac{x \sin kx}{k} + \frac{\cos kx}{k^2} + \text{常數} \quad 1A$$

$$(b) \quad \int_0^{\frac{\pi}{2}} x \cos x \cos 2x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} x (\cos 3x + \cos x) \, dx \quad 1M$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} x \cos 3x \, dx + \frac{1}{2} \int_0^{\frac{\pi}{2}} x \cos x \, dx$$

$$= \frac{1}{2} \left[\frac{x \sin 3x}{3} + \frac{\cos 3x}{9} \right]_0^{\frac{\pi}{2}} + \frac{1}{2} \left[x \sin x + \cos x \right]_0^{\frac{\pi}{2}} \quad 1M$$

$$= \frac{\pi}{6} - \frac{5}{9} \quad 1A$$

$$6. \quad (a) \quad \text{設 } u = \sin x. \text{ 則 } du = \cos x \, dx. \quad 1M$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} e^{\sin x} \cos^3 x \, dx &= \int_0^1 e^u (1 - u^2) \, du \\ &= \left[e^u (1 - u^2) \right]_0^1 + 2 \int_0^1 u e^u \, du \quad 1M \end{aligned}$$

$$= -1 + 2 \left[u e^u \right]_0^1 - 2 \int_0^1 e^u \, du$$

$$\begin{aligned} &= 1 + 2e - 2 \left[e^u \right]_0^1 \\ &= 1 \quad 1A \end{aligned}$$

$$(b) \quad \cos 3x = \cos(2x + x)$$

$$= \cos 2x \cos x - \sin 2x \sin x \quad 1M$$

$$= (2 \cos^2 x - 1) \cos x - (2 \sin x \cos x) \sin x \quad 1M$$

$$= 4 \cos^3 x - 3 \cos x \quad 1A$$

$$\int_0^{\frac{\pi}{2}} e^{\sin 3x} \, dx = \int_0^{\frac{\pi}{2}} e^{\sin x} (4 \cos^3 x - 3 \cos x) \, dx \quad 1M$$

$$= 4 \int_0^{\frac{\pi}{2}} e^{\sin x} \cos^3 x \, dx - 3 \int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx$$

$$= 4(1) - 3 \left[e^{\sin x} \right]_0^{\frac{\pi}{2}} \quad 1A$$

$$= 7 - 3e \quad 1A$$

7. (a) (i) $\tan \alpha = \frac{\cos \frac{2\pi}{13} - 1}{\sin \frac{2\pi}{13}}$

$$= \frac{1 - 2 \sin^2 \frac{\pi}{13} - 1}{2 \sin \frac{\pi}{13} \cos \frac{\pi}{13}}$$

$$= -\tan \frac{\pi}{13}$$

$$= \tan \left(-\frac{\pi}{13} \right)$$

$$\alpha = -\frac{\pi}{13}$$

(ii) $\tan \beta = \frac{\cos \frac{2\pi}{13} + 1}{\sin \frac{2\pi}{13}}$

$$= \frac{2 \cos^2 \frac{\pi}{13} - 1 + 1}{2 \sin \frac{\pi}{13} \cos \frac{\pi}{13}}$$

$$= \cot \frac{\pi}{13}$$

$$= \tan \left(\frac{\pi}{2} - \frac{\pi}{13} \right)$$

$$= \tan \frac{11\pi}{26}$$

$$\beta = \frac{11\pi}{26}$$

(b) (i) $x^2 + 2x \cos \frac{2\pi}{13} + 1 = \left(x + \cos \frac{2\pi}{13} \right)^2 + \sin^2 \frac{2\pi}{13}$

(ii) 設 $x + \cos \frac{2\pi}{13} = \sin \frac{2\pi}{13} \tan \theta$ 。則 $dx = \sin \frac{2\pi}{13} \sec^2 \theta d\theta$ 。

當 $x = -1$ ，

$$\tan \theta = \frac{-1 + \cos \frac{2\pi}{13}}{\sin \frac{2\pi}{13}}$$

$$\theta = -\frac{\pi}{13}$$

當 $x = 1$ ，

$$\tan \theta = \frac{1 + \cos \frac{2\pi}{13}}{\sin \frac{2\pi}{13}}$$

$$\theta = \frac{11\pi}{26}$$

$$\int_{-1}^1 \frac{\sin \frac{2\pi}{13}}{x^2 + 2x \cos \frac{2\pi}{13} + 1} dx = \int_{-\frac{\pi}{13}}^{\frac{11\pi}{26}} \frac{\sin \frac{2\pi}{13}}{\left(x + \cos \frac{2\pi}{13} \right)^2 + \sin^2 \frac{2\pi}{13}} dx$$

$$= \int_{-\frac{\pi}{13}}^{\frac{11\pi}{26}} \frac{\sin^2 \frac{2\pi}{13} \sec^2 \theta}{\sin^2 \frac{2\pi}{13} \tan^2 \theta + \sin^2 \frac{2\pi}{13}} d\theta$$

$$= \int_{-\frac{\pi}{13}}^{\frac{11\pi}{26}} d\theta$$

$$= \left[\theta \right]_{-\frac{\pi}{13}}^{\frac{11\pi}{26}}$$

$$= \frac{\pi}{2}$$

(c) 設 $u = -x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned} \int_{-1}^1 \frac{\sin \frac{11\pi}{13}}{x^2 + 2x \cos \frac{11\pi}{13} + 1} dx &= \int_{-1}^1 \frac{\sin \frac{2\pi}{13}}{x^2 - 2x \cos \frac{2\pi}{13} + 1} dx & 1A \\ &= - \int_1^{-1} \frac{\sin \frac{2\pi}{13}}{u^2 + 2u \cos \frac{2\pi}{13} + 1} du \\ &= \int_{-1}^1 \frac{\sin \frac{2\pi}{13}}{u^2 + 2u \cos \frac{2\pi}{13} + 1} du \\ &= \frac{\pi}{2} & 1A \end{aligned}$$

8. (a) $\int x(x+2)e^x dx = (x^2 + 2x)e^x - \int (2x + 2)e^x dx$ 1M

$$= (x^2 + 2x)e^x - 2(x+1)e^x + 2 \int e^x dx$$
 1M

$$\begin{aligned} &= (x^2 - 2)e^x + 2e^x + \text{常數} \\ &= x^2 e^x + \text{常數} \end{aligned}$$
 1A

(b) $\int_1^e x(x+2)e^x \ln x dx = \left[x^2 e^x \ln x \right]_1^e - \int_1^e x^2 e^x \cdot \frac{1}{x} dx$ 1M+1A

$$\begin{aligned} &= e^{e+2} - \int_1^e x e^x dx \\ &= e^{e+2} - \left[x e^x \right]_1^e + \int_1^e e^x dx \end{aligned}$$
 1M

$$\begin{aligned} &= e^{e+2} - (e^{e+1} - e) + \left[e^x \right]_1^e \\ &= e^{e+2} - e^{e+1} + e^e \end{aligned}$$
 1A

9. (a) $\frac{A}{x+1} + \frac{B}{x+3} = \frac{A(x+3) + B(x+1)}{(x+1)(x+3)}$
 $= \frac{(A+B)x + (3A+B)}{(x+1)(x+3)}$

可得 $A+B=1$ 及 $3A+B=0$ 。 1M

求解後，可得 $A = -\frac{1}{2}$ 及 $B = \frac{3}{2}$ 。 1A

(b) $\int \frac{x}{(x+1)(x+3)} dx = \int \left(-\frac{1}{2(x+1)} + \frac{3}{2(x+3)} \right) dx$ 1M

$$= -\frac{1}{2} \ln|x+1| + \frac{3}{2} \ln|x+3| + \text{常數}$$
 1A

(c) 設 $u = \sqrt{x}$ 。則 $du = \frac{1}{2\sqrt{x}} dx$ 。 1M

$$\begin{aligned} \int_1^9 \frac{1}{(\sqrt{x}+1)(\sqrt{x}+3)} dx &= 2 \int_1^9 \frac{\sqrt{x}}{2\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+3)} dx \\ &= 2 \int_1^3 \frac{u}{(u+1)(u+3)} du \\ &= 2 \left[-\frac{1}{2} \ln|u+1| + \frac{3}{2} \ln|u+3| \right]_1^3 & 1M \\ &= 3 \ln 3 - 4 \ln 2 & 1A \end{aligned}$$

$$10. \quad (a) \quad \int_{\frac{5\pi}{4}}^{2\pi} x \sin x \, dx = \left[-x \cos x \right]_{\frac{5\pi}{4}}^{2\pi} + \int_{\frac{5\pi}{4}}^{2\pi} \cos x \, dx \quad 1M$$

$$= -2\pi - \frac{5\sqrt{2}\pi}{8} + \left[\sin x \right]_{\frac{5\pi}{4}}^{2\pi} \quad 1A$$

$$= -\frac{(5\sqrt{2} + 16)\pi}{8} + \frac{\sqrt{2}}{2} \quad 1A$$

$$(b) \quad \int_{\frac{5\pi}{4}}^{2\pi} x^2 \sin x \, dx = \left[-x^2 \cos x \right]_{\frac{5\pi}{4}}^{2\pi} + 2 \int_{\frac{5\pi}{4}}^{2\pi} x \cos x \, dx \quad 1M$$

$$= -4\pi^2 - \frac{25\sqrt{2}\pi^2}{32} + 2 \left[x \sin x \right]_{\frac{5\pi}{4}}^{2\pi} - 2 \int_{\frac{5\pi}{4}}^{2\pi} \sin x \, dx \quad 1M$$

$$= -\frac{(25\sqrt{2} + 128)\pi^2}{32} + \frac{5\sqrt{2}\pi}{4} + 2 \left[\cos x \right]_{\frac{5\pi}{4}}^{2\pi}$$

$$= -\frac{(25\sqrt{2} + 128)\pi^2}{32} + \frac{5\sqrt{2}\pi}{4} + 2 + \sqrt{2} \quad 1A$$

$$(c) \quad \text{設 } u = e^x + \pi. \text{ 則 } du = e^x \, dx. \quad 1M$$

$$\int_{\ln \frac{\pi}{4}}^{\ln \pi} e^{2x} (e^x + \pi) \sin(e^x + \pi) \, dx$$

$$= \int_{\frac{5\pi}{4}}^{2\pi} (u - \pi) u \sin u \, du \quad 1A$$

$$= \int_{\frac{5\pi}{4}}^{2\pi} u^2 \sin u \, du - \pi \int_{\frac{5\pi}{4}}^{2\pi} u \sin u \, du$$

$$= -\frac{(25\sqrt{2} + 128)\pi^2}{32} + \frac{5\sqrt{2}\pi}{4} + 2 + \sqrt{2} - \pi \left[-\frac{(5\sqrt{2} + 16)\pi}{8} + \frac{\sqrt{2}}{2} \right] \quad 1M$$

$$= -\frac{(5\sqrt{2} + 64)\pi^2}{32} + \frac{3\sqrt{2}\pi}{4} + 2 + \sqrt{2} \quad 1A$$

$$11. \quad (a) \quad 2x^2 - x + 2 = 2\left(x - \frac{1}{4}\right)^2 + \frac{15}{8} \quad 1A$$

$$\text{設 } x - \frac{1}{4} = \frac{\sqrt{15}}{4} \tan \theta \circ \text{則 } dx = \frac{\sqrt{15}}{4} \sec^2 \theta d\theta \circ \quad 1M$$

$$\begin{aligned} \int \frac{1}{2x^2 - x + 2} dx &= \int \frac{1}{2\left(x - \frac{1}{4}\right)^2 + \frac{15}{8}} dx \\ &= \int \frac{\frac{\sqrt{15}}{4} \sec^2 \theta}{\frac{15}{8} \tan^2 \theta + \frac{15}{8}} d\theta \quad 1A \end{aligned}$$

$$\begin{aligned} &= \frac{2\sqrt{15}}{15} \int d\theta \\ &= \frac{2\sqrt{15}}{15} \theta + \text{常數} \\ &= \frac{2\sqrt{15}}{15} \tan^{-1} \frac{4x - 1}{\sqrt{15}} + \text{常數} \quad 1A \end{aligned}$$

$$(b) \quad \text{設 } u = \frac{1}{x} \circ \text{則 } du = -\frac{1}{x^2} dx \circ$$

$$\begin{aligned} \int_{\frac{1}{2}}^2 \frac{1}{(x^5 + 1)(2x^2 - x + 2)} dx &= \int_{\frac{1}{2}}^2 \frac{x^2}{(x^5 + 1)(2x^2 - x + 2)} dx \\ &= - \int_2^{\frac{1}{2}} \frac{\frac{1}{u^2}}{\frac{1}{u^5} + 1} \left(\frac{2}{u^2} - \frac{1}{u} + 2 \right) du \quad 1A \\ &= \int_{\frac{1}{2}}^2 \frac{u^5}{(u^5 + 1)(2u^2 - u + 2)} du \\ &= \int_{\frac{1}{2}}^2 \frac{x^5}{(x^5 + 1)(2x^2 - x + 2)} dx \end{aligned}$$

$$\begin{aligned} &\int_{\frac{1}{2}}^2 \frac{1}{(x^5 + 1)(2x^2 - x + 2)} dx \\ &= \frac{1}{2} \left[\int_{\frac{1}{2}}^2 \frac{1}{(x^5 + 1)(x^2 - x + 2)} dx + \int_{\frac{1}{2}}^2 \frac{x^5}{(x^5 + 1)(x^2 - x + 2)} dx \right] \\ &= \frac{1}{2} \int_{\frac{1}{2}}^2 \frac{1 + x^5}{(x^5 + 1)(x^2 - x + 2)} dx \\ &= \frac{1}{2} \int_{\frac{1}{2}}^2 \frac{1}{2x^2 - x + 2} dx \quad 1A \end{aligned}$$

$$= \frac{\sqrt{15}}{15} \left[\tan^{-1} \frac{4x - 1}{\sqrt{15}} \right]_{\frac{1}{2}}^2 \quad 1M$$

$$= \frac{\sqrt{15}}{15} \left(\tan^{-1} \frac{7\sqrt{15}}{15} - \tan^{-1} \frac{\sqrt{15}}{15} \right) \quad 1A$$

$$12. \quad (a) \quad \int e^{3x}(x^2 + 1) \, dx = \frac{e^{3x}(x^2 + 1)}{3} - \frac{2}{3} \int x e^{3x} \, dx \quad 1M$$

$$= \frac{e^{3x}(x^2 + 1)}{3} - \frac{2xe^{3x}}{9} + \frac{2}{9} \int e^{3x} \, dx \quad 1M$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{e^{3x}}{3} + \frac{2}{27} e^{3x} + \text{常數}$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{11e^{3x}}{27} + \text{常數} \quad 1A$$

(b) 設 $u = \tan \theta$ 。則 $du = \sec^2 \theta \, d\theta$ 。 1M

$$\int_0^{\frac{\pi}{4}} e^{3 \tan \theta} \sec^4 \theta \, d\theta = \int_0^{\frac{\pi}{4}} e^{3 \tan \theta} \sec^2 \theta (\tan^2 \theta + 1) \, d\theta \quad 1A$$

$$= \int_0^1 e^{3u} (u^2 + 1) \, du$$

$$= \left[\frac{u^2 e^{3u}}{3} - \frac{2ue^{3u}}{9} + \frac{11e^{3u}}{27} \right]_0^1 \quad 1M$$

$$= \frac{14e^3 - 11}{27} \quad 1A$$

13. (a) 設 $u = 1 - e^{-x}$ 。則 $du = e^{-x} \, dx$ 。 1M

$$\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}} \, dx = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{du}{u} \quad 1A$$

$$= \left[\ln |u| \right]_{\frac{1}{2}}^{\frac{2}{3}}$$

$$= \ln \frac{4}{3} \quad 1A$$

$$(b) \quad \frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1) + Bu}{u(u-1)}$$

$$= \frac{(A+B)u - A}{u(u-1)}$$

可得 $A + B = 0$ 及 $-A = 1$ 。 1M

求解後，可得 $A = -1$ 及 $B = 1$ 。 1A

$$(c) \quad \int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x - 1)} \, dx = \int_{\ln 2}^{\ln 3} \left(-\frac{1}{e^x} + \frac{1}{e^x - 1} \right) \, dx \quad 1M$$

$$= - \int_{\ln 2}^{\ln 3} e^{-x} \, dx + \int_{\ln 2}^{\ln 3} \frac{1}{e^x - 1} \, dx$$

$$= - \int_{\ln 2}^{\ln 3} e^{-x} \, dx + \int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}} \, dx$$

$$= \left[e^{-x} \right]_{\ln 2}^{\ln 3} + \ln \frac{4}{3} \quad 1M$$

$$= -\frac{1}{6} + \ln \frac{4}{3} \quad 1A$$