Dexter Wong & His Mathematics Team

暑期課程　2022 – 2023

數學　延伸部分

S5 – S6 M2 積分　功課　$\boxed{\text{第 5 套}}$

名字：________________

分校：________

上課日：日 / 一 / 二 / 三 / 四 / 五 / 六

考生須知

1. 本試卷**各題均須作答**，答案須寫在本試題答題簿中預留的空位內。
2. 除特別指明外，須詳細列出所有算式。
3. 除特別指明外，數值答案須用真確值表示。
4. 本試卷的附圖不一定依比例繪成。

建議題解

派發於暑期課程 S5 – S6 M2 積分 第 2 期 – 第 1 堂

1. (a) $\dfrac{A}{x+1}+\dfrac{B}{x+3}=\dfrac{A(x+3)+B(x+1)}{(x+1)(x+3)}$

$=\dfrac{(A+B)x+(3A+B)}{(x+1)(x+3)}$

可得 $A+B=1$ 及 $3A+B=0$。 1M

求解後，可得 $A=-\dfrac{1}{2}$ 及 $B=\dfrac{3}{2}$。 1A

(b) $\displaystyle\int \frac{x}{(x+1)(x+3)}\,dx=\int\left(-\frac{1}{2(x+1)}+\frac{3}{2(x+3)}\right)dx$ 1M

$=-\dfrac{1}{2}\ln|x+1|+\dfrac{3}{2}\ln|x+3|+$ 常數 1A

(c) 設 $u=\sqrt{x}$。則 $du=\dfrac{1}{2\sqrt{x}}\,dx$。 1M

$$\int_1^9 \frac{1}{(\sqrt{x}+1)(\sqrt{x}+3)}\,dx=2\int_1^9 \frac{\sqrt{x}}{2\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+3)}\,dx$$

$$=2\int_1^3 \frac{u}{(u+1)(u+3)}\,du$$

$$=2\left[-\frac{1}{2}\ln|u+1|+\frac{3}{2}\ln|u+3|\right]_1^3$$ 1M

$$=3\ln 3-4\ln 2$$ 1A

2. (a) $\dfrac{x}{x^2+4}-\dfrac{1}{x+4}=\dfrac{x(x+4)-(x^2+4)}{(x^2+4)(x+4)}$ 1M

$=\dfrac{4x-4}{(x^2+4)(x+4)}$

$=\dfrac{4(x-1)}{(x^2+4)(x+4)}$ 1

(b) 設 $t=2\tan\theta$。則 $dt=2\sec^2\theta\,d\theta$。

$$\int_0^{\frac{\pi}{4}} \frac{2\tan\theta-1}{\tan\theta+2}\,d\theta=\int_0^{\frac{\pi}{4}} \frac{2\tan\theta-1}{2\sec^2\theta(\tan\theta+2)}\cdot 2\sec^2\theta\,d\theta$$

$$=\int_0^2 \frac{t-1}{2\left(1+\frac{t^2}{4}\right)\left(\frac{t}{2}+2\right)}\,dt$$ 1A

$$=\int_0^2 \frac{4(t-1)}{(t^2+4)(t+4)}\,dt$$

$$=\int_0^2\left(\frac{t}{t^2+4}-\frac{1}{t+4}\right)dt$$ 1M

設 $u=t^2+4$。則 $du=2t\,dt$。 1M

$$\int_0^{\frac{\pi}{4}} \frac{2\tan\theta-1}{\tan\theta+2}\,d\theta=\frac{1}{2}\int_4^8 \frac{du}{u}-\int_0^2 \frac{dt}{t+4}$$

$$=\frac{1}{2}\Big[\ln|u|\Big]_4^8-\Big[\ln|t+4|\Big]_0^2$$ 1A

$$=\frac{1}{2}(\ln 8-\ln 4)-(\ln 6-\ln 2)$$

$$=\frac{5}{2}\ln 2-\ln 6$$ 1A

3. (a) $\displaystyle\int_4^2 \frac{2f(x)}{f(x)-2g(x)}\,dx = -2\int_2^4 \frac{f(x)}{f(x)-2g(x)}\,dx$

$= -2(10)$

$= -20$ 1A

(b) $\displaystyle\int_2^4 \frac{g(t)}{f(t)-2g(t)}\,dt = \int_2^4 \frac{g(x)}{f(x)-2g(x)}\,dx$

$\displaystyle= -\frac{1}{2}\int_2^4 \left[1-\frac{f(x)}{f(x)-2g(x)}\right]dx$ 1M

$\displaystyle= -\frac{1}{2}\Big[x\Big]_2^4 + \frac{1}{2}\int_2^4 \frac{f(x)}{f(x)-2g(x)}\,dx$

$\displaystyle= -\frac{1}{2}(4-2) + \frac{1}{2}(10)$

$= 4$ 1A

4. (a) $\dfrac{dy}{dx} = \tan^2 x\sec^2 x - \sec^2 x + 1$ 1M

$= \tan^2 x(1+\tan^2 x) - \tan^2 x$

$= \tan^4 x$ 1A

(b) $\displaystyle\int_0^{\frac{\pi}{6}} 3\tan^4 x\,dx = 3\int_0^{\frac{\pi}{6}} \tan^4 x\,dx$

$\displaystyle= 3\left[\frac{1}{3}\tan^3 4x - \tan x + x\right]_0^{\frac{\pi}{6}}$ 1M

$\displaystyle= \frac{\pi}{2} - \frac{8\sqrt{3}}{9}$ 1A

5. (a) 設 $x = \sin\theta$。則 $dx = \cos\theta\, d\theta$。

$\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$ 1M

$\displaystyle= \int d\theta$

$= \sin^{-1} x +$ 常數 1A

(b) $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \sqrt{\frac{1-x}{1+x}}\,dx = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \sqrt{\frac{(1-x)^2}{(1+x)(1-x)}}\,dx$ 1M

$\displaystyle= \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \left(\frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}\right)dx$

$\displaystyle= \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2}\int_{\frac{3}{4}}^{\frac{1}{2}} \frac{du}{\sqrt{u}}$ 其中 $u = 1 - x^2$ 1M

$\displaystyle= \Big[\sin^{-1} x\Big]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} + \frac{1}{2}\Big[2u^{\frac{1}{2}}\Big]_{\frac{3}{4}}^{\frac{1}{2}}$ 1M

$\displaystyle= \frac{\pi}{12} + \frac{\sqrt{2}-\sqrt{3}}{2}$ 1A

6. (a) 設 $u = 1 - e^{-x}$。則 $\mathrm{d}u = e^{-x}\,\mathrm{d}x$。 1M

$$\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}}\,\mathrm{d}x = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{\mathrm{d}u}{u} \qquad \text{1A}$$

$$= \Big[\ln|u|\Big]_{\frac{1}{2}}^{\frac{1}{3}}$$

$$= \ln\frac{4}{3} \qquad \text{1A}$$

(b) $$\frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1) + Bu}{u(u-1)}$$

$$= \frac{(A+B)u - A}{u(u-1)}$$

可得 $A + B = 0$ 及 $-A = 1$。 1M

求解後，可得 $A = -1$ 及 $B = 1$。 1A

(c) $$\int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x - 1)}\,\mathrm{d}x = \int_{\ln 2}^{\ln 3} \left(-\frac{1}{e^x} + \frac{1}{e^x - 1}\right)\mathrm{d}x \qquad \text{1M}$$

$$= -\int_{\ln 2}^{\ln 3} e^{-x}\,\mathrm{d}x + \int_{\ln 2}^{\ln 3} \frac{1}{e^x - 1}\,\mathrm{d}x$$

$$= -\int_{\ln 2}^{\ln 3} e^{-x}\,\mathrm{d}x + \int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}}\,\mathrm{d}x$$

$$= \Big[e^{-x}\Big]_{\ln 2}^{\ln 3} + \ln\frac{4}{3} \qquad \text{1M}$$

$$= -\frac{1}{6} + \ln\frac{4}{3} \qquad \text{1A}$$

7. (a) $$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xe^{\frac{1}{x}} + x^2e^x\left(-\frac{1}{x^2}\right) \qquad \text{1M}$$

$$= (2x - 1)e^{\frac{1}{x}} \qquad \text{1A}$$

(b) $$\int_1^2 \left[(2x-1)e^{\frac{1}{x}} - e^x\right]\mathrm{d}x = \int_1^2 (2x-1)e^{\frac{1}{x}}\,\mathrm{d}x - \int_1^2 e^x\,\mathrm{d}x$$

$$= \Big[x^2e^{\frac{1}{x}}\Big]_1^2 - \Big[e^x\Big]_1^2 \qquad \text{1M}$$

$$= 4e^{\frac{1}{2}} - e^2 \qquad \text{1A}$$

8. $$\int_0^2 \frac{x^3 - 27}{x - 3}\,\mathrm{d}x = \int_0^2 \frac{(x-3)(x^2 + 3x + 9)}{x - 3}\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_0^2 (x^2 + 3x + 9)\,\mathrm{d}x$$

$$= \left[\frac{x^3}{3} + \frac{3x^2}{2} + 9x\right]_0^2 \qquad \text{1M}$$

$$= \frac{80}{3} \qquad \text{1A}$$

9. (a) $2^3 - 6(2)^2 + 2k - 4 = 0$ 1M

$k = 10$ 1A

(b) $f(x) = x^3 - 6x^2 + 10x - 4$

$= (x-2)(x^2 - 4x + 2)$ 1A

(c) $\displaystyle\int_3^6 \frac{f(x)}{x-2}\,dx = \int_3^6 (x^2 - 4x + 2)\,dx$ 1M

$= \left[\frac{x^3}{3} - 2x^2 + 2x\right]_3^6$ 1M

$= 15$ 1A

10. $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin^2 3x}{1+\cos 2x}\,dx + \int_{\frac{\pi}{3}}^0 \frac{\sin^2 3x - 2}{1+\cos 2x}\,dx = \int_0^{\frac{\pi}{3}} \frac{\sin^2 3x}{1+\cos 2x}\,dx + \int_0^{\frac{\pi}{3}} \frac{2-\sin^2 3x}{1+\cos 2x}\,dx$ 1M

$\displaystyle = 2\int_0^{\frac{\pi}{3}} \frac{dx}{1+\cos 2x}\,dx$

$\displaystyle = 2\int_0^{\frac{\pi}{3}} \frac{dx}{2\cos^2 x}\,dx$

$\displaystyle = \int_0^{\frac{\pi}{3}} \sec^2 x\,dx$ 1A

$= \left[\tan x\right]_0^{\frac{\pi}{3}}$ 1M

$= \sqrt{3}$ 1A

11. (a) $y = 5^{x+1}$

$= e^{(x+1)\ln 5}$ 1M

$\dfrac{dy}{dx} = e^{(x+1)\ln 5}(\ln 5)$ 1M

$= 5^{x+1}\ln 5$ 1A

(b) $\displaystyle\int_1^3 5^{x+1}\,dx = \left[\frac{5^{x+1}}{\ln 5}\right]_1^3$ 1M

$= \dfrac{600}{\ln 5}$ 1A

12. 設 $u = x^2 + 3$。則 $du = 2x\,dx$。 1M

$\displaystyle\int_1^3 \frac{x^2 - 2x + 3}{x^2+3}\,dx = \int_1^3 \left(1 - \frac{2x}{x^2+3}\right)dx$ 1M

$\displaystyle = \int_1^3 dx - \int_4^{12} \frac{du}{u}$ 1A

$= \left[x\right]_1^3 - \left[\ln|u|\right]_4^{12}$ 1M

$= 2 - \ln 3$ 1A

13. 設 $u = 2x + 1$。則 $\mathrm{d}u = 2\,\mathrm{d}x$。 1M

$$\int_0^4 \frac{x+2}{\sqrt{2x+1}}\,\mathrm{d}x = \frac{1}{2}\int_1^9 \frac{\frac{u-1}{2}+2}{\sqrt{u}}\,\mathrm{d}u$$

$$= \frac{1}{2}\int_1^9 \left(\frac{\sqrt{u}}{2} + \frac{3}{2\sqrt{u}}\right)\mathrm{d}u$$ 1M

$$= \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{3} + 3u^{\frac{1}{2}}\right]_1^9$$

$$= \frac{22}{3}$$ 1A