Dexter Wong & His Mathematics Team

暑期課程　2022 – 2023

數學　延伸部分

S5 – S6 M2 微分　功課　第 7 套

名字：________________　　　　分校：__________

上課日：日 / 一 / 二 / 三 / 四 / 五 / 六

考生須知

1. 本試卷**各題均須作答**，答案須寫在本試題答題簿中預留的空位內。
2. 除特別指明外，須詳細列出所有算式。
3. 除特別指明外，數值答案須用真確值表示。
4. 本試卷的附圖不一定依比例繪成。

建議題解

派發於暑期課程
S5 – S6 M2 微分
第 2 期 – 第 3 堂

1. (a) Let T hours be the time taken.

$$T = \frac{x}{8} + \frac{\sqrt{(6-x)^2 + 4^2}}{3} \quad \text{1A}$$
$$= \frac{x}{8} + \frac{\sqrt{x^2 - 12x + 52}}{3}$$

(b) $$\frac{dT}{dx} = \frac{1}{8} + \frac{1}{6}(x^2 - 12x + 52)^{-\frac{1}{2}}(2x - 12) \quad \text{1M}$$
$$= \frac{1}{8} + \frac{2x - 12}{6(x^2 - 12x + 52)^{\frac{1}{2}}}$$

When $\frac{dT}{dx} = 0$,

$$\frac{1}{8} + \frac{2x - 12}{6\sqrt{x^2 - 12x + 52}} = 0 \quad \text{1M}$$
$$(8x - 48)^2 = 3^2(x^2 - 12x + 52)$$
$$55x^2 - 660x + 1836 = 0$$
$$x = \frac{330 - 12\sqrt{55}}{55} \text{ or } \frac{330 + 12\sqrt{55}}{55} \text{ (rejected)} \quad \text{1A}$$

x	$0 < x < \frac{330 - 12\sqrt{55}}{55}$	$\frac{330 - 12\sqrt{55}}{55} < x < 6$
$\frac{dT}{dx}$	$-$	$+$

1M

T attains minimum when $x = \frac{330 - 12\sqrt{55}}{55} \approx 4.38$.

Minimum time taken $= \frac{x}{8} + \frac{\sqrt{x^2 - 12x + 52}}{3}$

$\approx 1.99\text{ h} > 1.5\text{ h}$ 1A

The claim is disagreed. 1A

2. (a) $32\pi = 2\pi rh + 2\pi r^2$ 1M

$$h = \frac{16}{r} - r \quad 1$$

(b) $$V = \pi r^2 h + \frac{2}{3}\pi r^3$$
$$= \pi r^2\left(\frac{16}{r} - r\right) + \frac{2}{3}\pi r^3 \quad \text{1M}$$
$$= \left(16r - \frac{1}{3}r^3\right)\pi \quad 1$$

(c) $$\frac{dV}{dr} = (16 - r^2)\pi$$

當 $\frac{dV}{dr} = 0$，$r = 4$ 或 -4（捨去） 1M

由於外表面面積為 32π cm^2，可得 $0 < r \le 4$。

由於對 $0 < r < 4$，$\frac{dV}{dr} > 0$，V 在 $r = 4$ 時達至極大值。 1M

極大容量 $= \left(16(4) - \frac{1}{3}(4)^3\right)\pi = \frac{128\pi}{3}$ cm^3 < 135 cm^3。 1A

該容量不可能大於 135 cm^3。 1A

3. (a) $\dfrac{x}{\sin\left(\pi - \frac{2\pi}{3} - \theta\right)} = \dfrac{3}{\sin\frac{2\pi}{3}}$ 1M

$$x = \dfrac{3\sin\left(\frac{\pi}{3} - \theta\right)}{\left(\frac{\sqrt{3}}{2}\right)}$$

$= 2\sqrt{3}\sin\left(\dfrac{\pi}{3} - \theta\right)$ 1A

(b) $y = \dfrac{1}{2}(x)(3)\sin\theta$

$= 3\sqrt{3}\sin\left(\dfrac{\pi}{3} - \theta\right)\sin\theta$ 1

(c) $y = \dfrac{3\sqrt{3}}{2}\left[\cos\left(\dfrac{\pi}{3} - 2\theta\right) - \cos\dfrac{\pi}{3}\right]$

$\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 3\sqrt{3}\sin\left(\dfrac{\pi}{3} - 2\theta\right)$ 1

(d) 當 $\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 0$，

$$\sin\left(\dfrac{\pi}{3} - 2\theta\right) = 0$$

$$\dfrac{\pi}{3} - 2\theta = 0$$

$\theta = \dfrac{\pi}{6}$ 1A

θ	$0 < \theta < \frac{\pi}{6}$	$\frac{\pi}{6} < \theta < \frac{\pi}{3}$
$\frac{\mathrm{d}y}{\mathrm{d}\theta}$	+	−

1M

y 在 $\theta = \dfrac{\pi}{6}$ 達至最大值。

所求之值 $= 3\sqrt{3}\sin\dfrac{\pi}{6}\sin\dfrac{\pi}{6} = \dfrac{3\sqrt{3}}{4}$ 1A

4. (a) (i) $AE = AC - CE$

$1\cot\theta = 8\cos\theta - x$ 1M

$x = 8\cos\theta - \cot\theta$ 1A

(ii) $\dfrac{dx}{d\theta} = \csc^2\theta - 8\sin\theta$ 1A

當 $\dfrac{dx}{d\theta} = 0$，

$$\frac{1}{\sin^2\theta} = 8\sin\theta$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

θ	$0 < \theta < \dfrac{\pi}{6}$	$\dfrac{\pi}{6} < \theta < \dfrac{\pi}{2}$
$\dfrac{dx}{d\theta}$	+	−

1M

當 $\theta = \dfrac{\pi}{6}$ 時，x 達至極大值。 1

(b) (i) $S = \dfrac{1}{2}(8\sin\theta)(8\cos\theta) - \dfrac{1}{2}(1\cot\theta)(1)$ 1M

$= \dfrac{1}{2}(32\sin 2\theta - \cot\theta)$ 1

(ii) $\dfrac{dS}{d\theta} = 32\cos 2\theta + \dfrac{1}{2}\csc^2\theta$ 1A

當 $\theta = \dfrac{\pi}{6}$，$\dfrac{dS}{d\theta} = 18 \neq 0$。 1M

因此，當 x 達至極大值時，S 不同時達至極大值。 1A

5. (a) (i) $\dfrac{FG}{BC} = \dfrac{6-x}{6}$ 1M

$FG = \dfrac{2\sqrt{3}(6-\mathrm{x})}{3}$ cm

$S = \dfrac{1}{2}\left(\dfrac{2\sqrt{3}(6-x)}{3}\right)(x)$

$= \dfrac{\sqrt{3}x(6-x)}{3}$ 1

(ii) $\dfrac{\mathrm{d}S}{\mathrm{d}x} = \dfrac{\sqrt{3}}{3}(6-2x)$ 1A

當 $\dfrac{\mathrm{d}S}{\mathrm{d}x} = 0$，$x = 3$。 1M

x	$0 < x < 3$	$3 < x < 6$
$\dfrac{\mathrm{d}S}{\mathrm{d}x}$	$+$	$-$

1M

S 在 $x = 3$ 時達至極大值。 1A

(b) (i) $P = \dfrac{2\sqrt{3}(6-x)}{3} + 2\sqrt{x^2 + \left[\dfrac{\sqrt{3}(6-x)}{3}\right]^2}$ 1M

$= \dfrac{2\sqrt{3}(6-x)}{3} + 2\sqrt{x^2 + \dfrac{x^2 - 12x + 36}{3}}$

$= \dfrac{2\sqrt{3}(6-x)}{3} + 4\sqrt{\dfrac{x^2 - 3x + 9}{3}}$ 1A

(ii) $\dfrac{\mathrm{d}P}{\mathrm{d}x} = -\dfrac{2\sqrt{3}}{3} + \dfrac{2}{\sqrt{3}}(x^2 - 3x + 9)^{-\frac{1}{2}}(2x-3)$

$= -\dfrac{2\sqrt{3}}{3} + \dfrac{2(2x-3)}{\sqrt{3(x^2 - 3x + 9)}}$ 1A

當 $\dfrac{\mathrm{d}P}{\mathrm{d}x} = 0$，

$$\frac{2(2x-3)}{\sqrt{3(x^2 - 3x + 9)}} = \frac{2\sqrt{3}}{3}$$

$$(2x-3)^2 = x^2 - 3x + 9$$

$$3x^2 - 9x = 0$$

$x = 3$ 或 0（捨去）

x	$0 < x < 3$	$3 < x < 6$
$\dfrac{\mathrm{d}P}{\mathrm{d}x}$	$-$	$+$

1M

在 $x = 3$ 時，P 達至其極小值而 S 達至其極大值。

不同意該宣稱。 1A

6. (a) $100e^{-2.5} = [-(25)^2 + A(25) + B]e^{-\frac{25}{10}}$ 1M

$25A + B = 725$

$75e^{-3} = [-(30)^2 + A(30) + B]e^{-\frac{30}{10}}$

$30A + B = 975$

Solving, we have $A = 50$ and $B = -525$. 1A+1A

(b) (i) $\frac{\mathrm{d}W(x)}{\mathrm{d}x} = (-x^2 + 50x - 525)e^{-\frac{x}{10}}\left(-\frac{1}{10}\right) + e^{-\frac{x}{10}}(-2x + 50)$ 1M

$= \frac{1}{10}(x^2 - 70x + 1025)e^{-\frac{x}{10}}$ 1A

(ii) When $\frac{\mathrm{d}W(x)}{\mathrm{d}x} = 0$,

$x^2 - 70x + 1025 = 0$ 1M

$x = 35 - 10\sqrt{2}$ or $35 + 10\sqrt{2}$ (rejected) 1A

x	$15 < x < 35 - 10\sqrt{2}$	$35 - 10\sqrt{2} < x < 35$
$\frac{\mathrm{d}W(x)}{\mathrm{d}x}$	$+$	$-$

1M

$W(x)$ attains its maximum at $x = 35 - 10\sqrt{2}$.

Maximum weight $= [-(35 - 10\sqrt{2})^2 + 50(35 - 10\sqrt{2}) - 525]e^{-\frac{35-10\sqrt{2}}{10}}$

$\approx 10.29\,\mathrm{t}$ 1A

(c) $P(x) = \frac{1000W(x)}{x}$

$\frac{\mathrm{d}P(x)}{\mathrm{d}x} = 1000\left[\frac{xW'(x) - W(x)}{x^2}\right]$ 1M

$\left.\frac{\mathrm{d}P(x)}{\mathrm{d}x}\right|_{x=35-10\sqrt{2}} = 1000\left[\frac{(35 - 10\sqrt{2})W'(35 - 10\sqrt{2}) - W(35 - 10\sqrt{2})}{(35 - 10\sqrt{2})^2}\right]$

$= 1000\left[\frac{-W(35 - 10\sqrt{2})}{(35 - 10\sqrt{2})^2}\right]$

$\neq 0$ 1M

$P(x)$ does not attains its maximum when $W(x)$ attains its maximum.

The claim is incorrect. 1A

7. $f'(x) = \frac{1 - \ln x}{x^2}$ 1A

當 $f'(x) = 0$，

$1 - \ln x = 0$ 1M

$x = e$

x	1	e	4
$f(x)$	0	$\frac{1}{e}$	$\frac{\ln 4}{4}$

1M

由於 $0 < \frac{\ln 4}{4} < \frac{1}{e}$，

全局極大值 $= \frac{1}{e}$，全局極小值 $= 0$。 1A+1A

8. (a) (i) $a = \frac{(AB)(1)}{2} + \frac{(BC)(1)}{2} + \frac{(CA)(1)}{2}$ 1M

$2a = AB + BC + CA$

$s = 2a$ 1

(ii) $AB + BC + CA = 2a$

$x + x + 2x\sin\theta = 2 \times \frac{1}{2}(x)^2 \sin 2\theta$ 1M+1M

$2x(1+\sin\theta) = x^2 \sin 2\theta$

$$x = \frac{2(1+\sin\theta)}{\sin 2\theta}$$

$$s = 2 \times \frac{2(1+\sin\theta)}{\sin 2\theta} \times (1+\sin\theta)$$

$= \frac{4(1+\sin\theta)^2}{\sin 2\theta}$ 1

(b) $\frac{ds}{d\theta} = \frac{8(1+\sin\theta)(\cos\theta)(\sin 2\theta) - 4(1+\sin\theta)^2(2\cos 2\theta)}{\sin^2 2\theta}$ 1M

$$= \frac{8(1+\sin\theta)(\cos\theta\sin 2\theta - \sin\theta\cos 2\theta - \cos 2\theta)}{\sin^2 2\theta}$$

$$= \frac{8(1+\sin\theta)[\sin(2\theta-\theta) - \cos 2\theta]}{\sin^2 2\theta}$$

$= \frac{8(1+\sin\theta)(\sin\theta - \cos 2\theta)}{\sin^2 2\theta}$ 1A

當 $\frac{ds}{d\theta} = 0$，$\sin\theta = -1$（捨去）或 $\sin\theta - \cos 2\theta = 0$。

$$\sin\theta - (1 - 2\sin^2\theta) = 0$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$\sin\theta = \frac{1}{2}$ 或 -1（捨去）

$\theta = \frac{\pi}{6}$ 1A

θ	$0 < \theta < \frac{\pi}{6}$	$\frac{\pi}{6} < \theta < \frac{\pi}{2}$
$\frac{ds}{d\theta}$	$-$	$+$

1M

當 $\theta = \frac{\pi}{6}$ 時，周界為極小值。 1A

(c) $x = \frac{2(1+\sin\theta)}{\sin 2\theta}$

$\frac{dx}{d\theta} = \frac{2\cos\theta\sin 2\theta - 2(1+\sin\theta)(2\cos 2\theta)}{\sin^2 2\theta}$ 1M

當 $\theta = \frac{\pi}{6}$，

$\frac{ds}{d\theta} = \frac{2\sin\frac{\pi}{3}\cos\frac{\pi}{6} - 4\cos\frac{\pi}{3}\left(1+\sin\frac{\pi}{6}\right)}{\sin^2\frac{\pi}{3}} = -2 \neq 0$ 1M

x 的值不達至其極小值。

不同意該宣稱。 1A

9. (a) $N = 2PC + PA$ 1M

$= 2h\sec\theta + (50 - h\tan\theta)$ 1

(b) $N = 100\sec\theta + 50 - 50\tan\theta$

$\frac{dN}{d\theta} = 100\sec\theta\tan\theta - 50\sec^2\theta$ 1A

$= 50\sec^2\theta(2\sin\theta - 1)$

當 $\frac{dN}{d\theta} = 0$，

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$ 1A

θ	$0 < \theta < \frac{\pi}{6}$	$\frac{\pi}{6} < \theta < \frac{\pi}{2}$
$\frac{dN}{d\theta}$	$-$	$+$

1M

當 $\theta = \frac{\pi}{6}$ 時，N 達至極小值。

所求成本 $= 50\sec\frac{\pi}{6}\left(2\tan\frac{\pi}{6} - \sec\frac{\pi}{6}\right) = \$50(\sqrt{3} + 1)$ 1A

(c) (i) $\tan\theta = \frac{BP}{BC} < \frac{AB}{BC} = \frac{1}{\sqrt{3}}$ 1

當 $\tan\theta < \frac{1}{\sqrt{3}}$，$0 < \theta < \frac{\pi}{6}$ 及利用 (a)，$\frac{dN}{d\theta} < 0$。 1

(ii) 由於 $h = 100 > 50\sqrt{3}$，可得 $\frac{dN}{d\theta} < 0$。 1M

當 θ 為極大值時，N 為極小值，即該貨車直接向 A 駛去。 1A