

Dexter Wong & His Mathematics Team
Summer Course 2022 – 2023
MATHEMATICS Extended Part
S5 – S6 M2 Integration Assignment Set 4

Name: _____

Centre: _____

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings must be clearly shown.
3. Unless otherwise specified, numerical answers must be exact.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution



Distributed in summer course
S5 – S6 M2 Integration
Phase 1 – Lesson 4

$$1. \int x \cdot 7^x dx = \int x e^{x \ln 7} dx \quad 1M$$

$$= \frac{x e^{x \ln 7}}{\ln 7} - \frac{1}{\ln 7} \int e^{x \ln 7} dx \quad 1M$$

$$= \frac{x \cdot 7^x}{\ln 7} - \frac{7^x}{(\ln 7)^2} + \text{constant} \quad 1A$$

$$2. \int \sec^2 x \ln(\sin x) dx = \tan x \ln(\sin x) - \int \tan x \cdot \frac{\cos x}{\sin x} dx \quad 1M$$

$$= \tan x \ln(\sin x) - \int dx$$

$$= \tan x \ln(\sin x) - x + \text{constant} \quad 1A$$

$$3. \int e^{\sqrt{x}} dx = \int \sqrt{x} \cdot \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$\text{Let } u = \sqrt{x}. \text{ Then } du = \frac{1}{2\sqrt{x}} dx.$$

$$\int e^{\sqrt{x}} dx = 2 \int u \cdot e^u du$$

$$= 2ue^u - 2 \int e^u du \quad 1M$$

$$= 2ue^u - 2e^u + \text{constant}$$

$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + \text{constant} \quad 1A$$

$$4. \int x^2 \cos(3x - 4) dx = \frac{x^2 \sin(3x - 4)}{3} - \frac{2}{3} \int x \sin(3x - 4) dx \quad 1M$$

$$= \frac{x^2 \sin(3x - 4)}{3} + \frac{2x \cos(3x - 4)}{9} - \frac{2}{9} \int \cos(3x - 4) dx \quad 1M$$

$$= \frac{x^2 \sin(3x - 4)}{3} + \frac{2x \cos(3x - 4)}{9} - \frac{2 \sin(3x - 4)}{27} + \text{constant} \quad 1A$$

$$5. \int x^5 (\ln x)^2 dx = \frac{x^6 (\ln x)^2}{6} - \frac{1}{3} \int x^5 \ln x dx \quad 1M$$

$$= \frac{x^6 (\ln x)^2}{6} - \frac{x^6 \ln x}{18} + \frac{1}{18} \int x^6 dx \quad 1M$$

$$= \frac{x^6 (\ln x)^2}{6} - \frac{x^6 \ln x}{18} + \frac{x^6}{108} + \text{constant} \quad 1A$$

$$6. \int (x^2 + e^x) e^x dx = \int x^2 e^x dx + \int e^{2x} dx$$

$$= x^2 e^x - 2 \int x e^x dx + \int e^{2x} dx \quad 1M$$

$$= x^2 e^x - 2x e^x + 2 \int e^x dx + \int e^{2x} dx \quad 1M$$

$$= x^2 e^x - 2x e^x + 2e^x + \frac{1}{2} e^{2x} + \text{constant} \quad 1A$$

$$7. \quad \int e^{-3x} \cos 4x \, dx = -\frac{e^{-3x} \cos 4x}{3} - \frac{4}{3} \int e^{-3x} \sin 4x \, dx \quad 1M$$

$$= -\frac{e^{-3x} \cos 4x}{3} + \frac{4e^{-3x} \sin 4x}{9} - \frac{16}{9} \int e^{-3x} \cos 4x \, dx \quad 1M$$

$$\frac{25}{9} \int e^{-3x} \cos 4x \, dx = -\frac{e^{-3x} \cos 4x}{3} + \frac{4e^{-3x} \sin 4x}{9} + \text{constant} \quad 1M$$

$$\int e^{-3x} \cos 4x \, dx = -\frac{3e^{-3x} \cos 4x}{25} + \frac{4e^{-3x} \sin 4x}{25} + \text{constant} \quad 1A$$

$$8. \quad \int x^2 \cos(\ln x) \, dx = \frac{x^3 \cos(\ln x)}{3} + \frac{1}{3} \int x^2 \sin(\ln x) \, dx \quad 1M$$

$$= \frac{x^3 \cos(\ln x)}{3} + \frac{x^3 \sin(\ln x)}{9} - \frac{1}{9} \int x^2 \cos(\ln x) \, dx \quad 1M$$

$$\frac{10}{9} \int x^2 \cos(\ln x) \, dx = \frac{x^3 \cos(\ln x)}{3} + \frac{x^3 \sin(\ln x)}{9} + \text{constant} \quad 1M$$

$$\int x^2 \cos(\ln x) \, dx = \frac{3x^3 \cos(\ln x)}{10} + \frac{x^3 \sin(\ln x)}{10} + \text{constant} \quad 1A$$

$$9. \quad \text{Let } u = \sqrt{x}. \text{ Then } du = \frac{1}{2\sqrt{x}} \, dx.$$

$$\int \sin \sqrt{x} \, dx = 2 \int u \sin u \, du \quad 1M$$

$$= -2u \cos u + 2 \int \cos u \, du \quad 1M$$

$$= -2u \cos u + 2 \sin u + \text{constant} \quad 1A$$

$$= -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + \text{constant} \quad 1A$$

$$10. \quad \int x^2 \cos 3x \, dx = \frac{x^2 \sin 3x}{3} - \frac{2}{3} \int x \sin 3x \, dx \quad 1M+1A$$

$$= \frac{x^2 \sin 3x}{3} + \frac{2x \cos 3x}{9} - \frac{2}{9} \int \cos 3x \, dx \quad 1M$$

$$= \frac{x^2 \sin 3x}{3} + \frac{2x \cos 3x}{9} - \frac{2 \sin 3x}{27} + \text{constant} \quad 1A$$

$$11. \quad (a) \quad \text{Let } u = 1 + x^2. \text{ Then } du = 2x \, dx.$$

$$\int \frac{x^3}{1+x^2} \, dx = \frac{1}{2} \int \frac{u-1}{u} \, du \quad 1M$$

$$= \frac{1}{2} \int (1 - u^{-1}) \, du$$

$$= \frac{u}{2} - \frac{1}{2} \ln |u| + \text{constant}$$

$$= \frac{x^2}{2} - \frac{1}{2} \ln(1 + x^2) + \text{constant} \quad 1A$$

$$(b) \quad \int x^2 \tan^{-1} x \, dx = \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx \quad 1M+1A$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \ln(1 + x^2) + \text{constant} \quad 1A$$

12. (a) Let $2x = \tan \theta$. Then $2 dx = \sec^2 \theta d\theta$. 1M

$$\begin{aligned}\int \frac{1}{1+4x^2} dx &= \frac{1}{2} \int \frac{\sec^2 \theta}{1+\tan^2 \theta} d\theta & 1M \\ &= \frac{\theta}{2} + \text{constant} \\ &= \frac{1}{2} \tan^{-1} 2x + \text{constant} & 1A\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \int \ln(1+4x^2) dx &= x \ln(1+4x^2) - \int x \cdot \frac{8x}{1+4x^2} dx & 1M+1M \\ &= x \ln(1+4x^2) - 2 \int \left(1 - \frac{1}{1+4x^2}\right) dx & 1M \\ &= x \ln(1+4x^2) - 2x + \tan^{-1} 2x + \text{constant} & 1A\end{aligned}$$

$$\begin{aligned}13. \quad \int e^{2x} (\sin x + \cos x)^2 dx &= \int e^{2x} (1 + 2 \sin x \cos x) dx & 1M \\ &= \frac{e^{2x}}{2} + \int e^{2x} \sin 2x dx & 1M\end{aligned}$$

Consider the integral $\int e^{2x} \sin 2x dx$,

$$\begin{aligned}\int e^{2x} \sin 2x dx &= \frac{e^{2x}}{2} \sin 2x - \int \frac{e^{2x}}{2} (2 \cos 2x) dx & 1M \\ &= \frac{e^{2x} \sin 2x}{2} - \frac{e^{2x} \cos 2x}{2} + \int \frac{e^{2x}}{2} (-2 \sin 2x) dx & 1M\end{aligned}$$

$$2 \int e^{2x} \sin 2x dx = \frac{e^{2x} \sin 2x}{2} - \frac{e^{2x} \cos 2x}{2} + \text{constant} \quad 1M$$

$$\int e^{2x} \sin 2x dx = \frac{e^{2x}}{4} (\sin 2x - \cos 2x) + \text{constant} \quad 1A$$

$$\text{Therefore, } \int e^{2x} (\sin x + \cos x)^2 dx = \frac{e^{2x}}{2} + \frac{e^{2x}}{4} (\sin 2x - \cos 2x) + \text{constant} \quad 1A$$

14. (a) By long division,

$$\begin{aligned}\frac{x^7}{x^2+1} &= \frac{(x^5 - x^3 + x)(x^2 + 1) - x}{x^2 + 1} & 1M \\ &= x^5 - x^3 + x - \frac{x}{x^2 + 1} & 1A\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \int x^5 \ln(x^2 + 1) dx &= \frac{x^6 \ln(x^2 + 1)}{6} - \frac{1}{6} \int x^6 \times \frac{2x}{x^2 + 1} dx & 1M \\ &= \frac{x^6 \ln(x^2 + 1)}{6} - \frac{1}{3} \int \left(x^5 - x^3 + x - \frac{x}{x^2 + 1}\right) dx & 1M \\ &= \frac{x^6 \ln(x^2 + 1)}{6} - \frac{x^6}{18} + \frac{x^4}{12} - \frac{x^2}{6} + \frac{1}{6} \int \frac{d(x^2 + 1)}{x^2 + 1} & 1M \\ &= \frac{x^6 \ln(x^2 + 1)}{6} + \frac{\ln(x^2 + 1)}{6} - \frac{x^6}{18} + \frac{x^4}{12} - \frac{x^2}{6} + \text{constant} & 1A\end{aligned}$$

15. (a) Let $u = x^2 + 3x + 1$. Then $du = (2x + 3) dx$. 1M

$$\int (2x + 3)\sqrt{x^2 + 3x + 1} dx = \int \sqrt{u} du \quad 1A$$

$$= \frac{2u^{\frac{3}{2}}}{3} + \text{constant}$$

$$= \frac{2(x^2 + 3x + 1)^{\frac{3}{2}}}{3} + \text{constant} \quad 1A$$

(b) $\int \sin^4 x dx = \frac{1}{4} \int (1 - \cos 2x)^2 dx \quad 1M$

$$= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4} \int \left[1 - 2 \cos 2x + \frac{1 + \cos 4x}{2} \right] dx \quad 1M$$

$$= \frac{1}{8} \int (3 - 4 \cos 2x + \cos 4x) dx$$

$$= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + \text{constant} \quad 1A$$

(c) Let $u = x^2$. Then $du = 2x dx$. 1M

$$\int x^3 e^{x^2} dx = \frac{1}{2} \int u e^u du$$

$$= \frac{u e^u}{2} - \frac{1}{2} \int e^u du \quad 1M$$

$$= \frac{u e^u}{2} - \frac{e^u}{2} + \text{constant} \quad 1A$$

$$= \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + \text{constant} \quad 1A$$

16. (a) Let $2x = \tan \theta$. Then $2 \, dx = \sec^2 \theta \, d\theta$. 1M

$$\begin{aligned} \int \frac{dx}{1+4x^2} &= \frac{1}{2} \int \frac{\sec^2 \theta}{1+\tan^2 \theta} d\theta & 1A \\ &= \frac{\theta}{2} + \text{constant} \\ &= \frac{1}{2} \tan^{-1} 2x + \text{constant} & 1A \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int \frac{x^2}{1+4x^2} dx &= \frac{1}{4} \int \left(1 - \frac{1}{1+4x^2} \right) dx & 1M+1A \\ &= \frac{x}{4} - \frac{1}{8} \tan^{-1} 2x + \text{constant} & 1A \end{aligned}$$

(c) Let $x = \cos^2 \theta$. Then $dx = -2 \sin \theta \cos \theta \, d\theta$. 1M

$$\begin{aligned} &\int \sin 2\theta \ln(1+4\cos^4 \theta) d\theta \\ &= \int 2 \sin \theta \cos \theta \ln(1+4\cos^4 \theta) d\theta \\ &= - \int \ln(1+4x^2) dx & 1A \\ &= -x \ln(1+4x^2) + \int x \times \frac{(8x)}{1+4x^2} dx & 1M \\ &= -x \ln(1+4x^2) + 8 \times \left(\frac{x}{4} - \frac{1}{8} \tan^{-1} 2x \right) + \text{constant} & 1M \\ &= -x \ln(1+4x^2) + 2x - \tan^{-1} 2x + \text{constant} \\ &= -\cos^2 \theta \ln(1+4\cos^4 \theta) + 2\cos^2 \theta - \tan^{-1}(2\cos^2 \theta) + \text{constant} & 1A \end{aligned}$$