Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Extended Part

S5 – S6 M2 Integration Assignment $\boxed{\text{Set 3}}$

Name: ____________________ Centre: __________

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings must be clearly shown.
3. Unless otherwise specified, numerical answers must be exact.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution

Distributed in summer course
S5 – S6 M2 Integration
Phase 1 – Lesson 3

1. Let $x = e\sin\theta$. Then $\mathrm{d}x = e\cos\theta\,\mathrm{d}\theta$. 1M

$$\int x^2\sqrt{e^2-x^2}\,\mathrm{d}x = \int (e\sin\theta)^2\sqrt{e^2-e^2\sin^2\theta}\,e\cos\theta\,\mathrm{d}\theta$$

$$= e^4\int \sin^2\theta\cos^2\theta\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{e^4}{4}\int (2\sin\theta\cos\theta)^2\,\mathrm{d}\theta$$

$$= \frac{e^4}{4}\int \sin^2 2\theta\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{e^4}{8}\int (1-\cos 4\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{e^4\theta}{8} - \frac{e^4\sin 4\theta}{32} + \text{constant}$$

$$= \frac{e^4\theta}{8} - \frac{e^4}{16}\sin 2\theta\cos 2\theta + \text{constant}$$

$$= \frac{e^4\theta}{8} - \frac{e^4}{8}\sin\theta\cos\theta(1-2\sin^2\theta) + \text{constant}$$

$$= \frac{e^4\theta}{8} - \frac{e^4}{8}\left(\frac{x}{e}\right)\left(\frac{\sqrt{e^2-x^2}}{e}\right)\left[1-2\left(\frac{x}{e}\right)^2\right] + \text{constant}$$

$$= \frac{e^4}{8}\sin^{-1}\frac{x}{e} - \frac{1}{8}x(e^2-2x^2)\sqrt{e^2-x^2} + \text{constant} \qquad \text{1A}$$

2. (a) $4x - x^2 = 4 - (x-2)^2$ 1A

(b) Let $x - 2 = 2\sin\theta$. Then $\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\mathrm{d}x}{\sqrt{4x-x^2}} = \int \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,\mathrm{d}\theta$$

$$= \int \frac{\cos\theta}{\cos\theta}\,\mathrm{d}\theta$$

$$= \int \mathrm{d}\theta \qquad \text{1A}$$

$$= \theta + \text{constant}$$

$$= \sin^{-1}\frac{x-2}{2} + \text{constant} \qquad \text{1A}$$

3. $x^2 + 8x + 17 = (x+4)^2 + 1$

Let $x + 4 = \tan\theta$. Then $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{x}{x^2+8x+17}\,\mathrm{d}x = \int \frac{(\tan\theta - 4)\sec^2\theta}{\tan^2\theta + 1}\,\mathrm{d}\theta$$

$$= \int (\tan\theta - 4)\,\mathrm{d}\theta \qquad \text{1A}$$

Let $u = \cos\theta$. Then $\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{x}{x^2+8x+17}\,\mathrm{d}x = -\int \frac{\mathrm{d}u}{u} - \int 4\,\mathrm{d}\theta$$

$$= -\ln|\cos\theta| - 4\theta + \text{constant}$$

$$= -\ln\left|\frac{1}{\sqrt{x^2+8x+17}}\right| - 4\tan^{-1}(x+4) + \text{constant}$$

$$= \ln\sqrt{x^2+8x+17} - 4\tan^{-1}(x+4) + \text{constant} \qquad \text{1A}$$

4. Let $\ln x = \sin\theta$. Then $\frac{1}{x}\,\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\sqrt{1-(\ln x)^2}}{x}\,\mathrm{d}x = \int \sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta$$

$$= \int \cos^2\theta\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{constant}$$

$$= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} + \text{constant}$$

$$= \frac{1}{2}\sin^{-1}(\ln x) + \frac{1}{2}(\ln x)\sqrt{1-(\ln x)^2} + \text{constant} \qquad \text{1A}$$

5. Let $e^x = \tan\theta$. Then $e^x\,\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{e^x\,\mathrm{d}x}{(1+e^{2x})^2} = \int \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,\mathrm{d}\theta$$

$$= \int \cos^2\theta\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{constant}$$

$$= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + \text{constant}$$

$$= \frac{1}{2}\tan^{-1}e^x + \frac{1}{2}\left(\frac{e^x}{\sqrt{1+e^{2x}}}\right)\left(\frac{1}{\sqrt{1+e^{2x}}}\right) + \text{constant}$$

$$= \frac{1}{2}\tan^{-1}e^x + \frac{e^x}{2(1+e^{2x})} + \text{constant} \qquad \text{1A}$$

6. Let $x = \sqrt{3}\sin\theta$. Then $\mathrm{d}x = \sqrt{3}\cos\theta\,\mathrm{d}\theta$. 1M

$$\int \sqrt{1-\frac{x^2}{3}}\,\mathrm{d}x = \frac{1}{\sqrt{3}}\int \sqrt{3-3\sin^2\theta}\cdot\sqrt{3}\cos\theta\,\mathrm{d}\theta$$

$$= \sqrt{3}\int \cos^2\theta\,\mathrm{d}\theta \quad \text{1A}$$

$$= \frac{\sqrt{3}}{2}\int (1+\cos 2\theta)\,\mathrm{d}\theta \quad \text{1M}$$

$$= \frac{\sqrt{3}\theta}{2} + \frac{\sqrt{3}}{4}\sin 2\theta + \text{constant}$$

$$= \frac{\sqrt{3}\theta}{2} + \frac{\sqrt{3}}{2}\sin\theta\cos\theta + \text{constant}$$

$$= \frac{\sqrt{3}}{2}\sin^{-1}\frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{2}\left(\frac{x}{\sqrt{3}}\right)\left(\frac{\sqrt{3-x^2}}{\sqrt{3}}\right) + \text{constant}$$

$$= \frac{\sqrt{3}}{2}\sin^{-1}\frac{x}{\sqrt{3}} + \frac{x\sqrt{3(3-x^2)}}{6} + \text{constant} \quad \text{1A}$$

7. (a) $$x^2 + 2x\sin\frac{5\pi}{12} + 1 = x^2 + 2x\sin\frac{5\pi}{12} + \sin^2\frac{5\pi}{12} + \cos^2\frac{5\pi}{12}$$

$$= \left(x + \sin\frac{5\pi}{12}\right)^2 + \cos^2\frac{5\pi}{12} \quad \text{1A}$$

(b) Let $x + \sin\frac{5\pi}{12} = \cos\frac{5\pi}{12}\tan\theta$. Then $\mathrm{d}x = \cos\frac{5\pi}{12}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\cos\frac{5\pi}{12}}{x^2 + 2x\sin\frac{5\pi}{12} + 1}\,\mathrm{d}x = \int \frac{\cos\frac{5\pi}{12}}{\cos^2\frac{5\pi}{12}\tan^2\theta + \cos^2\frac{5\pi}{12}}\cdot\cos\frac{5\pi}{12}\sec^2\theta\,\mathrm{d}\theta$$

$$= \int \mathrm{d}\theta \quad \text{1A}$$

$$= \theta + \text{constant}$$

$$= \tan^{-1}\left(\frac{x + \sin\frac{5\pi}{12}}{\cos\frac{5\pi}{12}}\right) + \text{constant} \quad \text{1A}$$

$$= \tan^{-1}\left(\frac{x\csc\frac{5\pi}{12} + 1}{\cot\frac{5\pi}{12}}\right) + \text{constant}$$

$$= \tan^{-1}\frac{x(\sqrt{6}-\sqrt{2}) + 1}{2-\sqrt{3}} + \text{constant} \quad \text{1A}$$

8. (a) $x^2 + 2x\cos\dfrac{7\pi}{12} + 1 = x^2 + 2x\cos\dfrac{7\pi}{12} + \sin^2\dfrac{7\pi}{12} + \cos^2\dfrac{7\pi}{12}$

$$= \left(x + \cos\frac{7\pi}{12}\right)^2 + \sin^2\frac{7\pi}{12} \qquad \text{1A}$$

(b) Let $x + \cos\dfrac{7\pi}{12} = \sin\dfrac{7\pi}{12}\tan\theta$. Then $\mathrm{d}x = \sin\dfrac{7\pi}{12}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}
\int \frac{1}{x^2 + 2x\cos\frac{7\pi}{12} + 1}\,\mathrm{d}x &= \int \frac{\sin\frac{7\pi}{12}\sec^2\theta}{\sin^2\frac{7\pi}{12}\tan^2\theta + \sin^2\frac{7\pi}{12}}\,\mathrm{d}\theta \\
&= \csc\frac{7\pi}{12}\int \mathrm{d}\theta && \text{1A} \\
&= \csc\frac{7\pi}{12}\theta + \text{constant} \\
&= \csc\frac{7\pi}{12}\tan^{-1}\left(x\csc\frac{7\pi}{12} + \cot\frac{7\pi}{12}\right) + \text{constant} && \text{1A} \\
&= (\sqrt{6} - \sqrt{2})\tan^{-1}\left[(\sqrt{6} - \sqrt{2})x + \sqrt{3} - 2\right] + \text{constant} && \text{1A}
\end{aligned}$$

9. (a) $x^2 + 2x + 5 = (x + 1)^2 + 4$ 1A

(b) Let $x + 1 = 2\tan\theta$. Then $\mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}
\int \frac{\mathrm{d}x}{(x + 1)^2(x^2 + 2x + 5)} &= \int \frac{2\sec^2\theta}{(2\tan\theta)^2(4\tan^2\theta + 4)}\,\mathrm{d}\theta && \text{1M+1A} \\
&= \frac{1}{8}\int \cot^2\theta\,\mathrm{d}\theta \\
&= \frac{1}{8}\int (\csc^2\theta - 1)\,\mathrm{d}\theta && \text{1M} \\
&= -\frac{1}{8}\cot\theta - \frac{\theta}{8} + \text{constant} \\
&= -\frac{1}{8}\left(\frac{2}{x + 1}\right) - \frac{1}{8}\tan^{-1}\frac{x + 1}{2} + \text{constant} \\
&= -\frac{1}{4(x + 1)} - \frac{1}{8}\tan^{-1}\frac{x + 1}{2} + \text{constant} && \text{1A}
\end{aligned}$$

10. (a) Let $u = \tan x$. Then $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$. 1M

$$\int \sec^4 x \tan^3 x\,\mathrm{d}x = \int (u^2+1)u^3\,\mathrm{d}u \quad \text{1M}$$
$$= \int (u^5+u^3)\,\mathrm{d}u$$
$$= \frac{u^6}{6} + \frac{u^4}{4} + \text{constant}$$
$$= \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + \text{constant} \quad \text{1A}$$

(b) Let $x = 5\tan\theta$. Then $\mathrm{d}x = 5\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{1}{x^2\sqrt{25+x^2}}\,\mathrm{d}x = \int \frac{5\sec^2\theta}{25\tan^2\theta\sqrt{25+25\tan^2\theta}}\,\mathrm{d}\theta$$
$$= \frac{1}{25}\int \frac{\sec\theta}{\tan^2\theta}\,\mathrm{d}\theta$$
$$= \frac{1}{25}\int \frac{\cos\theta}{\sin^2\theta}\,\mathrm{d}\theta \quad \text{1M}$$

Let $w = \sin\theta$. Then $\mathrm{d}w = \cos\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{1}{x^2\sqrt{25+x^2}}\,\mathrm{d}x = \frac{1}{25}\int \frac{\mathrm{d}w}{w^2}$$
$$= -\frac{1}{25w} + \text{constant}$$
$$= -\frac{\sqrt{25+x^2}}{25x} + \text{constant} \quad \text{1A}$$

11. (a) Let $x = \sin^2\theta$. Then $\mathrm{d}x = 2\sin\theta\cos\theta\,\mathrm{d}\theta$

$$\int \frac{f(x)}{\sqrt{x(1-x)}}\,\mathrm{d}x = \int \frac{f(\sin^2\theta)\cdot 2\sin\theta\cos\theta}{\sqrt{\sin^2\theta(1-\sin^2\theta)}}\,\mathrm{d}\theta$$
$$= 2\int f(\sin^2\theta)\,\mathrm{d}\theta \quad \text{1}$$

(b) $$\int \frac{\mathrm{d}x}{\sqrt{x(1-x)}} = 2\int \mathrm{d}\theta$$
$$= 2\theta + \text{constant}$$
$$= 2\sin^{-1}\sqrt{x} + \text{constant} \quad \text{1A}$$

$$\int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \int \frac{x}{\sqrt{x(1-x)}}\,\mathrm{d}x$$
$$= 2\int \sin^2\theta\,\mathrm{d}\theta \quad \text{1A}$$
$$= \int (1-\cos 2\theta)\,\mathrm{d}\theta$$
$$= \theta - \frac{1}{2}\sin 2\theta + \text{constant} \quad \text{1A}$$
$$= \sin^{-1}\sqrt{x} - \sqrt{x(1-x)} + \text{constant} \quad \text{1A}$$

12. (a) Let $x = a\tan\theta$. Then $dx = a\sec^2\theta\, d\theta$.

$$\int \frac{1}{x^2+a^2}\,dx = \int \frac{a\sec^2\theta}{a^2\tan^2\theta + a^2}\,d\theta \qquad \text{1M+1A}$$

$$= \frac{1}{a}\theta + \text{constant}$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + \text{constant} \qquad \text{1}$$

(b) (i) $t = \tan\frac{\theta}{2}$

$$1 = \sec^2\frac{\theta}{2} \times \frac{1}{2}\frac{d\theta}{dt} \qquad \text{1A}$$

$$\frac{d\theta}{dt} = \frac{2}{1+\tan^2\frac{\theta}{2}}$$

$$= \frac{2}{1+t^2} \qquad \text{1}$$

(ii) Let $t = \tan\frac{\theta}{2}$. Then $d\theta = \frac{2}{1+t^2}\,dt$.

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$.

$$\int \frac{1}{2+\cos\theta}\,d\theta = \int \frac{\frac{2}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}\,dt \qquad \text{1M}$$

$$= \int \frac{2}{2+2t^2+1-t^2}\,dt$$

$$= 2\int \frac{1}{t^2+3}\,dt$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + \text{constant} \qquad \text{1M}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{\tan\frac{\theta}{2}}{\sqrt{3}} + \text{constant} \qquad \text{1A}$$

13. (a) (i) $\dfrac{A}{u^2}+\dfrac{B}{u-1}+\dfrac{C}{u+1}=\dfrac{A(u+1)(u-1)+Bu^2(u+1)+Cu^2(u-1)}{u^2(u-1)(u+1)}$

Therefore, we have $1\equiv A(u-1)(u+1)+Bu^2(u+1)+Cu^2(u-1)$. 1M

Put $u=0$, then $A=-1$; put $u=1$, then $B=\dfrac{1}{2}$; put $u=-1$, then $C=-\dfrac{1}{2}$. 1A+1A

(ii)
$$\int\frac{\mathrm{d}u}{u^2(u-1)(u+1)}\,\mathrm{d}u=\int\left(-\frac{1}{u^2}+\frac{1}{2(u-1)}-\frac{1}{2(u+1)}\right)\mathrm{d}u$$
$$=\frac{1}{u}+\frac{1}{2}\ln|u-1|-\frac{1}{2}\ln|u+1|+\text{constant} \qquad \text{1M+1A}$$
$$=\frac{1}{u}+\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+\text{constant}$$

(Remarks) pp -1 if $\boxed{C}$ is used.

C is used in the question and cannot be used to represent constant.

(b) (i) Let $x=3\tan\theta$. Then $\mathrm{d}x=3\sec^2\theta\,\mathrm{d}\theta$.
$$\int\frac{\sqrt{x^2+9}}{x}\,\mathrm{d}x=\int\frac{\sqrt{9\tan^2\theta+9}}{3\tan\theta}\times3\sec^2\theta\,\mathrm{d}\theta \qquad \text{1A}$$
$$=3\int\frac{\sec^3\theta}{\tan\theta}\,\mathrm{d}\theta$$
$$=3\int\frac{\mathrm{d}\theta}{\sin\theta\cos^2\theta} \qquad \text{1}$$

(ii) Let $u=\cos\theta$. Then $\mathrm{d}u=-\sin\theta\,\mathrm{d}\theta$.
$$\int\frac{\sqrt{x^2+9}}{x}\,\mathrm{d}x=3\int\frac{\mathrm{d}\theta}{\sin\theta\cos^2\theta}$$
$$=3\int\frac{-\sin\theta}{(\cos\theta+1)(\cos\theta-1)\cos^2\theta}\,\mathrm{d}\theta$$
$$=3\int\frac{\mathrm{d}u}{u^2(u+1)(u-1)} \qquad \text{1M}$$
$$=\frac{3}{u}+\frac{3}{2}\ln\left|\frac{u-1}{u+1}\right|+\text{constant} \qquad \text{1M}$$
$$=3\div\frac{3}{\sqrt{x^2+9}}+\frac{3}{2}\ln\left|\frac{\frac{3}{\sqrt{x^2+9}}-1}{\frac{3}{\sqrt{x^2+9}}+1}\right|+\text{constant}$$
$$=\sqrt{x^2+9}+\frac{3}{2}\ln\left|\frac{3-\sqrt{x^2+9}}{3+\sqrt{x^2+9}}\right|+\text{constant} \qquad \text{1A}$$