Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Extended Part

S5 – S6 M2 Differentiation Assignment $\boxed{\text{Set 4}}$

Name: ______________________ Centre: ____________

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings must be clearly shown.
3. Unless otherwise specified, numerical answers must be exact.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution

Distributed in summer course
S5 – S6 M2 Differentiation
Phase 1 – Lesson 4

1. (a) $\dfrac{dy}{dx} = 1 + 2\cos 2x$ 1A

$\dfrac{d^2y}{dx^2} = -4\sin 2x$ 1A

(b) When $\dfrac{dy}{dx} = 0$,

$1 + 2\cos 2x = 0$ 1M

$\cos 2x = -\dfrac{1}{2}$

$2x = \dfrac{2\pi}{3}$ or $\dfrac{4\pi}{3}$

$x = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$ 1A

When $x = \dfrac{\pi}{3}$, $\dfrac{d^2y}{dx^2} = -2\sqrt{3} < 0$. So, y attains maximum when $x = \dfrac{\pi}{3}$. 1A

Maximum value of $y = \dfrac{\pi}{3} + \sin\dfrac{2\pi}{3} = \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$ 1A

When $x = \dfrac{2\pi}{3}$, $\dfrac{d^2y}{dx^2} = 2\sqrt{3} > 0$. So, y attains minimum when $x = \dfrac{2\pi}{3}$.

Minimum value of $y = \dfrac{2\pi}{3} + \sin\dfrac{4\pi}{3} = \dfrac{2\pi}{3} - \dfrac{\sqrt{3}}{2}$. 1A

2. (a) $\ln f(x) = x\ln x - 6\ln(2x + 13)$ 1A

$\dfrac{f'(x)}{f(x)} = \ln x + x \cdot \dfrac{1}{x} - 6 \cdot \dfrac{2}{2x + 13}$ 1M+1A

$f'(x) = \left(\ln x + 1 - \dfrac{12}{2x + 13}\right) f(x)$

$= \left(\ln x + \dfrac{2x + 1}{2x + 13}\right) \dfrac{x^x}{(2x + 13)^6}$ 1A

(b) For $x > 1$, we have $\ln x > 0$, $\dfrac{2x + 1}{2x + 13} > 0$ and $\dfrac{x^x}{(2x + 13)^6} > 0$. 1M

So, $f'(x) > 0$ and hence $f(x)$ is an increasing function. 1

3. $f'(x) = 2\cos x - 1$ 1A

When $f'(x) = 0$,

$\cos x = \dfrac{1}{2}$

$x = \dfrac{\pi}{3}$ 1A

x	$0 < x < \dfrac{\pi}{3}$	$\dfrac{\pi}{3} < x < \pi$
$f'(x)$	$+$	$-$

1M

$f(0) = 0$; $f\left(\dfrac{\pi}{3}\right) = \sqrt{3} - \dfrac{\pi}{3}$ and $f(\pi) = -\pi$.

The greatest value of $f(x)$ is $\sqrt{3} - \dfrac{\pi}{3}$ 1A

The least value of $f(x)$ is $-\pi$. 1A

4. (a) $\dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{[(x+h)^3 - 3(x+h)] - (x^3 - 3x)}{h}$ 1M

$= \lim\limits_{h \to 0} \dfrac{(x^3 + 3hx^2 + 3h^2x + h^3) - (3x + 3h) - (x^3 - 3x)}{h}$ 1M

$= \lim\limits_{h \to 0} (3x^2 + 3xh + h^2 - 3)$

$= 3x^2 - 3$ 1A

(b) When C is decreasing, $\dfrac{dy}{dx} \leq 0$.

$3x^2 - 3 \leq 0$ 1M

$-1 \leq x \leq 1$ 1A

5. (a) $\dfrac{dy}{dx} = \cos x + 2\sin x$ 1A

$\dfrac{d^2y}{dx^2} = -\sin x + 2\cos x$ 1A

(b) When $\dfrac{dy}{dx} = 0$,

$\tan x = -\dfrac{1}{2}$

$x = \pi - \tan^{-1}\dfrac{1}{2}$ or $2\pi - \tan^{-1}\dfrac{1}{2}$ 1M

When $x = \pi - \tan^{-1}\dfrac{1}{2}$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{\sqrt{5}} - \dfrac{4}{\sqrt{5}} < 0$. 1M

When $x = 2\pi - \tan^{-1}\dfrac{1}{2}$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{\sqrt{5}} + \dfrac{4}{\sqrt{5}} > 0$

y attains its minimum at $x = 2\pi - \tan^{-1}\dfrac{1}{2}$.

Minimum value of $y = -\sin\left(\tan^{-1}\dfrac{1}{2}\right) - 2\cos\left(\tan^{-1}\dfrac{1}{2}\right) = -\dfrac{1}{\sqrt{5}} - \dfrac{4}{\sqrt{5}} = -\sqrt{5}$ 1M+1A

6. (a) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{[2(x+h)^3 - (x+h)^2] - [2x^3 - x^2]}{h}$ 1M

$= \lim\limits_{h\to 0} \dfrac{2(x^3 + 3x^2h + 3xh^2 + h^3) - (x^2 + 2xh + h^2) - 2x^3 + x^2}{h}$ 1M

$= \lim\limits_{h\to 0} \dfrac{6x^2h + 6xh^2 + 2h^3 - 2xh - h^2}{h}$

$= \lim\limits_{h\to 0} (6x^2 + 6xh + 2h^2 - 2x - h)$

$= 6x^2 - 2x$ 1A

(b) $6x^2 - 2x \geq 0$ 1M

$x \leq 0$ or $x \geq \dfrac{1}{3}$ 1A

(c) $\left.\dfrac{dy}{dx}\right|_{(-2,\ -20)} = 6(-2)^2 - 2(-2) = 28$ 1M

Required equation is

$y + 20 = 28(x + 2)$

$28x - y + 36 = 0$ 1A

7. (a) $f'(x) = 12x^2 + 2mx + n$

$$\begin{cases} -33 = 4(6)^3 + m(6)^2 + n(6) + 615 & \text{1M} \\ 0 = 12(6)^2 + 2m(6) + n & \text{1M} \end{cases}$$

Solving, $m = -30$ and $n = -72$. 1A

(b) When $f'(x) = 0$,

$12x^2 - 60x - 72 = 0$

$x = 6$ or -1 1M

$f''(x) = 24x - 60$

$f''(6) = 24(6) - 60 = 84 > 0$ 1M

$f''(-1) = -84 < 0$

When $x = -1$, $y = 653$.

Thus, the minimum value is -33 and the maximum value is 653. 1A

8. (a) $\dfrac{dy}{dx} = \lim_{h\to 0}\dfrac{\frac{(x+h)^2}{(x+h)+1} - \frac{x^2}{x+1}}{h}$ 1M

$= \lim_{h\to 0}\dfrac{(x+h)^2(x+1) - x^2(x+h+1)}{h(x+1)(x+h+1)}$

$= \lim_{h\to 0}\dfrac{hx^2 + (h^2+2h)x + h^2}{h(x+1)(x+h+1)}$

$= \lim_{h\to 0}\dfrac{x^2 + (h+2)x + h}{(x+1)(x+h+1)}$ 1M

$= \dfrac{x(x+2)}{(x+1)^2}$ 1A

(b) $\dfrac{x(x+2)}{(x+1)^2} \geq 0$ 1M

$x(x+2) \geq 0$

$x \leq -2$ or $x \geq 0$ 1A

(Accept $x < -2$ or $x > 0$)

9. (a) Slope of the given line $= \dfrac{-2}{4} = -\dfrac{1}{2}$.

Therefore, $f'(x) = -\dfrac{x}{2} + 2$ 1A

Required slope $= f'(1) = -\dfrac{1}{2} + 2 = \dfrac{3}{2}$. 1M+1A

(b) There is only one turning point, with x-coordinate = 4. 1A

It is a maximum point as the $f'(x)$ changes from positive to negative. 1A

10. (a) $f'(x) = \dfrac{1}{x^2} - \dfrac{\ln x}{x^2}$ 1A

When $f'(x) = 0$,

$\ln x = 1$

$x = e$ 1M

x	$0 < x < e$	$x > e$
$f'(x)$	$+$	$-$

1M

The maximum value of $f(x)$ is $f(e) = \dfrac{1}{e}$. 1A

(b) By (a), $f(x) \leq f(e)$.

$\dfrac{\ln x}{x} \leq \dfrac{1}{e}$ 1M

$\ln x^e \leq x$

$e^x \geq x^e$ 1

11. (a) $\frac{dy}{dx} = 3\sin^2\left(x + \frac{\pi}{3}\right)\cos\left(x + \frac{\pi}{3}\right)$ 1M

When $\frac{dy}{dx} = 0$, $x = \frac{\pi}{6}$.

The stationary point is $\left(\frac{\pi}{6},\ 1 + \pi\right)$. 1A

(b)

x	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{\pi}{2}$
$\frac{dy}{dx}$	$+$	$-$

1M

$\left(\frac{\pi}{6},\ 1 + \pi\right)$ is a maximum point. 1A

12. (a) $\frac{dy}{dx} = e^{x+1} + xe^{x+1}$ 1A

$= e^{x+1}(1 + x)$

When $\frac{dy}{dx} = 0$, $x = -1$.

Stationary point is $(-1,\ -1)$. 1A

(b)

x	$x < -1$	$x > -1$
$\frac{dy}{dx}$	$-$	$+$

1M

$(-1,\ -1)$ is a minimum point. 1A

13. $\frac{dy}{dx} = 2x(1 - 3x^2)^{\frac{1}{2}} + x^2\left(\frac{1}{2}\right)(1 - 3x^2)^{-\frac{1}{2}}(-6x)$ 1M

$= (1 - 3x^2)^{-\frac{1}{2}}\left[2x(1 - 3x^2) - 3x^3\right]$

$= (1 - 3x^2)^{-\frac{1}{2}}(-9x^3 + 2x)$

When $\frac{dy}{dx} = 0$, $x = 0$ or $\pm\frac{\sqrt{2}}{3}$.

x	$-\frac{\sqrt{3}}{3} < x < -\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3} < x < 0$	$0 < x < \frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3} < x < \frac{\sqrt{3}}{3}$
$\frac{dy}{dx}$	$+$	$-$	$+$	$-$

1M

Maximum points are $\left(-\frac{\sqrt{2}}{3},\ \frac{2\sqrt{3}}{27}\right)$ and $\left(\frac{\sqrt{2}}{3},\ \frac{2\sqrt{3}}{27}\right)$. 1A

Minimum point is $(0,\ 0)$. 1A

14. (a) $\frac{dy}{dx} = 3x^2 - 4kx - 4k^2$ 1M

$= (x - 2k)(3x + 2k)$

$\frac{d^2y}{dx^2} = 6x - 4k$

When $\frac{dy}{dx} = 0$, $x = 2k$ or $-\frac{2k}{3}$.

$\left.\frac{d^2y}{dx^2}\right|_{x=2k} = 6(2k) - 4k = 8k > 0$ 1M

Thus, y attains its local minimum at $x = 2k$. 1

(b) $(2k)^3 - 2k(2k)^2 - 4k^2(2k) = -1000$

$-8k^3 = -1000$

$k^3 = 125$

$k = 5$ 1A

15. (a) $f'(x) = 12x^2 + 4kx + 15$ 1A

(b) Since $f(x)$ is continuous and there are no turning points, we have $f'(x) \geq 0$ or $f'(x) \leq 0$ for all $x \in \mathbb{R}$.

$\Delta = (4k)^2 - 4(12)(15) \leq 0$ 1M

$16k^2 - 720 \leq 0$

$-3\sqrt{5} \leq k \leq 3\sqrt{5}$ 1A

16. (a) $\frac{dy}{dx} = \lim_{h \to 0} \frac{-\frac{6}{(x+h)^2} + \frac{6}{x^2}}{h}$ 1M

$= \lim_{h \to 0} \frac{6(x+h)^2 - 6x^2}{hx^2(x+h)^2}$

$= \lim_{h \to 0} \frac{12x + 6h}{x^2(x+h)^2}$ 1M

$= \frac{12x}{x^4}$

$= \frac{12}{x^3}$ 1A

(b) When the curve is increasing, $\frac{dy}{dx} = \frac{12}{x^3} \geq 0$. 1M

Required range is $x > 0$. 1A

17. (a) $y = 1 + \dfrac{\ln x}{x}$

$\dfrac{dy}{dx} = \dfrac{\frac{1}{x}(x) - \ln x(1)}{x^2}$ 1M

$= \dfrac{1 - \ln x}{x^2}$ 1A

$\dfrac{d^2y}{dx^2} = \dfrac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4}$

$= \dfrac{-3x + 2x \ln x}{x^4}$

$= \dfrac{2\ln x - 3}{x^3}$ 1A

(b) When $\dfrac{dy}{dx} = 0$,

$1 - \ln x = 0$

$\ln x = 1$

$x = e$

When $x = e$, $y = \dfrac{e + \ln e}{e} = \dfrac{e+1}{e}$.

Stationary point is $\left(e, \dfrac{e+1}{e}\right)$. 1A

$\left.\dfrac{d^2y}{dx^2}\right|_{x=e} = \dfrac{2\ln e - 3}{e^3} = -\dfrac{1}{e^3} < 0$ 1M

$\left(e, \dfrac{e+1}{e}\right)$ is a maximum point. 1A

18. (a) $\dfrac{dy}{dx} = \dfrac{2x(x+1) - (x^2 + p)}{(x+1)^2}$ 1M

$= \dfrac{x^2 + 2x - p}{(x+1)^2}$ 1A

(b) (i) $\dfrac{1^2 + 2 - p}{(1+1)^2} = 0$ 1M

$p = 3$ 1A

(ii) $x^2 + 2x - 3 = 0$ 1M

$x = -3$ or 1

x	$x < -3$	$-3 < x < -1$	$-1 < x < 1$	$x > 1$
$\dfrac{dy}{dx}$	+	−	−	+

1M

When $x = -3$, $y = -6$; when $x = 1$, $y = 2$.

The local maximum is -6 and the local minimum is 2. 1A+1A

19. (a) $f'(x) = 3x^2 + 2ax + b$ 1A

α and β are roots of $f'(x) = 0$ 1M

Thus, $\alpha + \beta = -\dfrac{2a}{3}$ and $\alpha\beta = \dfrac{b}{3}$. 1A

(b) Since $f'(x) = 0$ has two real roots,

$$\Delta = (2a)^2 - 4(3b) > 0$$ 1M

$$4a^2 - 12b > 0$$

$$a^2 > 3b$$ 1

(c) $$\frac{f(\alpha) - f(\beta)}{\alpha - \beta} = \frac{(\alpha^3 + a\alpha^2 + b\alpha + c) - (\beta^3 + a\beta^2 + b\beta + c)}{\alpha - \beta}$$

$$= \frac{(\alpha^3 - \beta^3) + a(\alpha^2 - \beta^2) + b(\alpha - \beta)}{\alpha - \beta}$$

$$= (\alpha^2 + \alpha\beta + \beta^2) + a(\alpha + \beta) + b$$ 1M

$$= (\alpha + \beta)^2 - \alpha\beta + a(\alpha + \beta) + b$$

$$= \frac{4a^2}{9} - \frac{b}{3} - \frac{2a^2}{3} + b$$ 1M

$$= \frac{2}{9}(3b - a^2)$$ 1

(d) Since $a^2 > 3b$, we have $\dfrac{2}{9}(3b - a^2) < 0$. So, $\dfrac{f(\alpha) - f(\beta)}{\alpha - \beta} < 0$ 1M

So, $f(\alpha) - f(\beta) < 0$ if $\alpha - \beta > 0$

That is, $f(\alpha) < f(\beta)$ if $\alpha > \beta$. 1A