Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Extended Part

S5 – S6 M2 Differentiation Assignment Set 3

Name: _____ Centre: _____

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings must be clearly shown.
3. Unless otherwise specified, numerical answers must be exact.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution

Distributed in summer course
S5 – S6 M2 Differentiation
Phase 1 – Lesson 3

1. $x \ln y^2 + y^2 = 9$

$\ln y^2 + \frac{2x}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$ 1M+1A

$\frac{dy}{dx} = -\frac{\ln y^2}{\frac{2x}{y} + 2y}$

$= -\frac{y \ln y^2}{2x + 2y^2}$

When $x = 0$, $y = \pm 3$.

At $(0,\ 3)$, $\frac{dy}{dx} = -\frac{3 \ln 9}{18} = -\frac{\ln 3}{3}$; at $(0,\ -3)$, $\frac{dy}{dx} = \frac{\ln 3}{3}$. 1A

At $(0,\ 3)$, the equation of tangent is $y = -\frac{x \ln 3}{3} + 3$. 1A

At $(0,\ -3)$, the equation of tangent is $y = \frac{x \ln 3}{3} - 3$. 1A

2. (a) $x^2 + xy + y^2 = 4$

$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$ 1M

$(x + 2y)\frac{dy}{dx} = -2x - y$

$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$ 1A

(b) At $(2,\ 0)$,

$\frac{dy}{dx} = \frac{-2(2) - 0}{2 + 0}$ 1M

$= -2$

Required equation is

$y - 0 = -2(x - 2)$ 1M

$2x + y - 4 = 0$ 1A

3. (a) $\frac{dy}{dx} = \frac{3}{2}(2x + 8)^{\frac{1}{2}}(2) + 6x$ 1M

$= 3\sqrt{2x + 8} + 6x$ 1A

(b) Slope of tangent $= -6$.

$3\sqrt{2x + 8} + 6x = -6$ 1M+1A

$\sqrt{2x + 8} = -2 - 2x$

$2x + 8 = 4x^2 + 8x + 4$

$4x^2 + 6x - 4 = 0$ 1M

$x = -2$ or $\frac{1}{2}$ (rejected) 1A

There is only one tangent to C parallel to the straight line $6x + y + 4 = 0$.

Thus, the claim is disagreed. 1A

4. Let $(a,\ b)$ be the point of contact.

Then $b = a^2 - 3a + 3$.

$$y = x^2 - 3x + 3$$

$$\frac{dy}{dx} = 2x - 3$$

Consider the line joining $(a,\ b)$ and $(3,\ 2)$.

$$\frac{b-2}{a-3} = 2a - 3 \qquad \text{1M}$$

$$\frac{(a^2 - 3a + 3) - 2}{a - 3} = 2a - 3 \qquad \text{1M}$$

$$a^2 - 3a + 1 = 2a^2 - 9a + 9$$

$$0 = a^2 - 6a + 8$$

$$a = 2 \quad \text{or} \quad 4$$

The point of contacts are $(2,\ 1)$ and $(4,\ 7)$. 1A

At $(2,\ 1)$, the equation of tangent is

$$y - 1 = [2(2) - 3](x - 2) \qquad \text{1M}$$

$$x - y - 1 = 0 \qquad \text{1A}$$

At $(4,\ 7)$, the equation of tangent is

$$y - 7 = [2(4) - 3](x - 4)$$

$$5x - y - 13 = 0 \qquad \text{1A}$$

5. (a) Selling price $= x(1 + 70\%)(1 - 5\%) = 1.615x$ 1A

$$\text{Percentage} = \frac{1.615x - x}{x} \times 100\% \qquad \text{1M}$$

$$= 61.5\% \qquad \text{1A}$$

(b) $1.615x = 2907$ 1M

$x = 1800$ 1A

6. $y^2 = x^2y + 2$

$2y\dfrac{dy}{dx} = 2xy + x^2\dfrac{dy}{dx}$ 1M

When $x = 1$,

$y^2 = y + 2$

$y = 2 \quad \text{or} \quad -1$

At $(1,\ 2)$, $\dfrac{dy}{dx} = \dfrac{4}{3}$ and the equation of tangent is 1A

$y - 2 = \dfrac{4}{3}(x - 1)$

$4x - 3y + 2 = 0$ 1A

At $(1,\ -1)$, $\dfrac{dy}{dx} = \dfrac{2}{3}$ and the equation of tangent is

$y + 1 = \dfrac{2}{3}(x - 1)$

$2x - 3y - 5 = 0$ 1A

7. $xe^y - y^3\cos x = 5$

$e^y + xe^y\dfrac{dy}{dx} - 3y^2\cos x\dfrac{dy}{dx} + y^3\sin x = 0$ 1M+1M

$(xe^y - 3y^2\cos x)\dfrac{dy}{dx} = -e^y - y^3\sin x$

$\dfrac{dy}{dx} = \dfrac{e^y + y^3\sin x}{3y^2\cos x - xe^y}$

When $y = 0$, $x = 5$. 1A

At $(5,\ 0)$, $\dfrac{dy}{dx} = \dfrac{1 + 0}{0 - 5} = -\dfrac{1}{5}$. 1M

Required equation is

$y - 0 = -\dfrac{1}{5}(x - 5)$

$x + 5y - 5 = 0$ 1A

8. $xy = 1$

$y + x\dfrac{dy}{dx} = 0$ 1M

$\dfrac{dy}{dx} = -\dfrac{y}{x}$

Let $(a,\ b)$ be a point on the curve. Then $ab = 1$ and

$\dfrac{b+4}{a-2} = -\dfrac{b}{a}$ 1M

$ab + 4a = 2b - ab$

$4a + 1 = 2b - 1 \quad (\text{since } ab = 1)$

$\dfrac{4}{b} + 1 = 2b - 1$ 1M

$0 = 2b^2 - 2b - 4$

$b = 2 \quad \text{or} \quad -1$ 1A

When $b = 2$, $a = \dfrac{1}{2}$. The equation of tangent is

$y - 2 = -4\left(x - \dfrac{1}{2}\right)$

$y = -4x + 4$ 1A

When $b = -1$, $a = -1$. The equation of tangent is

$y + 1 = -1(x + 1)$

$y = -x - 2$ 1A

9. Let the point of contact be $(a,\ b)$. Then $b^2 = 4a$.

$y^2 = 4x$

$2y\dfrac{dy}{dx} = 4$ 1M

$\dfrac{dy}{dx} = \dfrac{2}{y}$

Therefore,

$\dfrac{b-0}{a+1} = \dfrac{2}{b}$ 1M

$b^2 = 2a + 2$

$4a = 2a + 2$ 1M

$a = 1$

When $a = 1$, $b = \pm 2$.

The equations of tangents are

$y - 2 = \dfrac{2}{2}(x - 1) \quad \text{and} \quad y + 2 = \dfrac{2}{-2}(x - 1)$

$y = x + 1 \qquad\qquad y = -x - 1$ 1A+1A

10. (a) $y = ae^{\ln x^x}$

$= ae^{x \ln x}$

$\dfrac{dy}{dx} = ae^{x \ln x}\left(\ln x + x\left(\dfrac{1}{x}\right)\right)$ 1M

$= ax^x(\ln x + 1)$

$-6e^e = ae^e(\ln e + 1)$

$a = -3$ 1A

(b) When $x = 1$, $y = -3$.

$\left.\dfrac{dy}{dx}\right|_{x=1} = -3(1)(\ln 1 + 1) = -3$ 1M

Required equation is

$y + 3 = -3(x - 1)$

$3x + y = 0$ 1A

11. $x \ln y + y = 2$

$\ln y + x \cdot \dfrac{1}{y}\dfrac{dy}{dx} + \dfrac{dy}{dx} = 0$ 1M+1M

$\dfrac{dy}{dx} = -\dfrac{y \ln y}{x + y}$

When the curve cuts the y-axis, $x = 0$ and so $y = 2$. 1A

$\left.\dfrac{dy}{dx}\right|_{(0,\ 2)} = -\dfrac{2 \ln 2}{0 + 2} = -\ln 2$ 1M

The equation of the tangent is $y = -x \ln 2 + 2$. 1A

12. (a) $y = 8(3x - 2)^{-2}$

$\dfrac{dy}{dx} = -48(3x - 2)^{-3}$ 1A

Put $\dfrac{dy}{dx} = -\dfrac{3}{4}$,

$-\dfrac{3}{4} = -48(3x - 2)^{-3}$ 1M

$3x - 2 = \sqrt[3]{64}$

$x = 2$

The coordinates of P are $\left(2,\ \dfrac{1}{2}\right)$. 1A

(b) Required equation is

$y - \dfrac{1}{2} = \dfrac{4}{3}(x - 2)$ 1M

$8x - 6y - 13 = 0$ 1A

13. Let the coordinates of P be $(a,\ b)$.

$\left.\dfrac{dy}{dx}\right|_P = 3a^2$ 1A

Consider the slope of L,

$3a^2 = \dfrac{-16-b}{0-a}$ 1M

$3a^3 = b + 16$

$3a^3 = a^3 + 16$ 1M

$a = 2$

When $a = 2$, $b = 8$. 1A

The equation of L is $y = 12x - 16$. 1A

14. $xe^0 + 0 = 6$

$x = 6$ 1A

$xe^y + y^2 \sin x = 6$

$e^y + xe^y \dfrac{dy}{dx} + 2y \sin x \dfrac{dy}{dx} + y^2 \cos x = 0$ 1M+1M

$(xe^y + 2y \sin x)\dfrac{dy}{dx} = -e^y - y^2 \cos x$

$\dfrac{dy}{dx} = -\dfrac{e^y + y^2 \cos x}{xe^y + 2y \sin x}$

$\left.\dfrac{dy}{dx}\right|_{(6,\ 0)} = -\dfrac{e^0 + 0}{6e^0 + 0} = -\dfrac{1}{6}$ 1M

Required equation is

$y - 0 = -\dfrac{1}{6}(x - 6)$

$x + 6y - 6 = 0$ 1A

15. (a) $f'(x) = \dfrac{2}{\sqrt{x}} - 1$ 1M+1A

$\left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{2}{\sqrt{1}} - 1 = 1$

Required equation is

$y - 8 = 1(x - 1)$

$x - y + 7 = 0$ 1A

(b) $\dfrac{dy}{dx} = -\dfrac{1}{x-3}$ 1M

(1) $\left(-\dfrac{1}{x-3}\right) = -1$ 1M

$x = 4$

When $x = 4$, $y = 11$. The coordinates of Q are $(4,\ 11)$. 1A

16. When $x = 0$, $\ln y = 0$ and so $y = 1$. 1A

$$x^2 + xy = \ln y$$

$$2x + y + x\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$ 1M+1A

At $(0,\ 1)$,

$$0 + 1 + 0 = \frac{dy}{dx}$$

$$\frac{dy}{dx} = 1$$ 1A

The equation of tangent is $y = x + 1$. 1A