Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Compulsory Part

S5 – S6 Core Assignment Set 4

Name: ____________________ Centre: ____________

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings (except for multiple choice questions) must be clearly shown.
3. Unless otherwise specified, numerical answers should be either exact or correct to 3 significant figures.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution

Distributed in summer course
S5 – S6 Core
Phase 1 – Lesson 4

1. (a) $1+3=-\frac{p}{1}$ 1M

$p=-4$ 1A

$1\times 3=\frac{q}{1}$

$q=3$ 1A

(b) $y=x^2-4x+3$

$=(x^2-4x+4)-1$ 1M

$=(x-2)^2-1$ 1A

(c) $y=(2x+1)^2$

$=4\left(x+\frac{1}{2}\right)^2$

Reduce along the y-axis to $\frac{1}{4}$ times the original. 1A

Translate rightwards by $\frac{5}{2}$ units. 1A

Translate downwards by 1 unit. 1A

2. (a) $f(x)=\frac{x^2}{16}-\frac{x}{2}+11$

$=\frac{1}{16}(x^2-8x+16)+10$ 1M

$=\frac{1}{16}(x-4)^2+10$

The coordinates of vertex are $(4,\ 10)$. 1A

(b) $g(x)=-f(x)=-\frac{1}{16}(x-4)^2-10$ 1M

$h(x)=-\frac{1}{16}(x-4)^2+6$ 1M

y-intercept $=-\frac{1}{16}(0-4)^2+6=5$ 1A

3. (a) $f(x)=x^2-6kx+12k^2+6$

$=(x^2-6x+9k^2)+3k^2+6$ 1M

$=(x-3k)^2+3k^2+6$

The coordinates of vertex are $\left(3k,\ 3k^2+6\right)$. 1A

(b) $P\left(3k-3,\ 3k^2+6\right)$ and $Q\left(3k+3,\ -3k^2-6\right)$. 1A

Consider the the distances from point $(0,\ 6)$ to P and to Q.

$\sqrt{(3k-3)^2+(3k^2)^2}=\sqrt{(3k+3)^2+(-3k^2-6-6)^2}$ 1M

$-72k^2-36k-144=0$

$\Delta=36^2-4(-72)(-144)=-40\,176<0$ 1M

The equation has no real roots.

Thus, such point R does not exist. 1A

4. (a) $g(x) = x^2 - 2kx + 2k^2 + 4$

$= (x^2 - 2kx + k^2) + k^2 + 4$ 1M

$= (x - k)^2 + k^2 + 4$

Coordinates of vertex are $\left(k,\ k^2 + 4\right)$. 1A

(b) Coordinates of D and E are $\left(k - 2,\ k^2 + 4\right)$ and $\left(k + 2,\ -k^2 - 4\right)$ respectively. 1A

Suppose $(0,\ 3)$ is the circumcentre of $\triangle DEF$.

$$\sqrt{(k-2)^2 + (k^2 + 4 - 3)^2} = \sqrt{(k+2)^2 + (-k^2 - 4 - 3)^2}$$ 1M

$$(k-2)^2 - (k+2)^2 = (k^2 + 7)^2 - (k^2 + 1)^2$$

$$-8k = 12k^2 + 48$$

$$0 = 12k^2 + 8k + 48$$

$\Delta = 8^2 - 4(12)(48) = -2240 < 0$ 1M

The equation has no real roots. It is not possible for $(0,\ 3)$ to be the circumcentre.

Point F does not exist. 1A

5. (a) $g(x) = f(-x) = -x^3 - 2x^2 + 3x + 4$. 1A

(b) $h(x) = g(x + 1)$

$= -(x+1)^3 - 2(x+1)^2 + 3(x+1) + 4$

$= -x^3 - 5x^2 - 4x + 4$ 1A

(c) Let the resulting graph of translating $y = f(x)$ to the left by 1 unit and then reflecting along the y-axis be $y = k(x)$.

$k(x) = f(-x + 1)$ 1M

$= -x^3 + x^2 + 4x$ 1A

$\neq h(x)$

The claim is incorrect. 1A

6. (a) When $y = 0$, $3x^2 - 6mx + 4m^2 = 0$.

$$\Delta = (6m)^2 - 4(3)(4m^2) \qquad \text{1M}$$
$$= -12m^2$$
$$< 0$$

The graph has no x-intercepts. 1

(b) $y = f(x)$

$= 3(x^2 - 2mx + m^2) + m^2$ 1M

$= 3(x - m)^2 + m^2$

The coordinates of the vertex are $\left(m,\ m^2\right)$. 1A

(c) Coordinates of A and B are $\left(m,\ m^2\right)$ and $(m,\ 0)$ respectively. 1A

Since $\angle OBA = 90°$, the circumcentre is the midpoint of OA.

The coordinates of circumcentre are $\left(\frac{m}{2},\ \frac{m^2}{2}\right)$. 1A

When $m = 2$, the coordinates of circumcentre are $(1,\ 2)$ which does not lie on $y = x$. 1M

The claim is incorrect. 1A

7. (a) $f(x) = \frac{-1}{3}x^2 - \frac{k}{2}x + k - 1$

$= \frac{-1}{3}\left(x^2 + \frac{3k}{2} + \frac{9k^2}{16}\right) + \frac{3k^2}{16} + k - 1$ 1M

$= \frac{-1}{3}\left(x + \frac{3k}{4}\right)^2 + \frac{3k^2}{16} + k - 1$

The coordinates of S are $\left(-\frac{3k}{4}, \frac{3k^2}{16} + k - 1\right)$. 1A

(b) (i) $g(x) = \frac{3}{2}f(-x)$

The coordinates of T are $\left(\frac{3k}{4}, \frac{9k^2}{32} + \frac{3k}{2} - \frac{3}{2}\right)$. 1A

(ii) Since S is the orthocentre of $\triangle OST$, $\angle OST = 90°$.

$$\frac{\frac{3k^2}{16} + k - 1}{-\frac{3k}{4}} \times \frac{\frac{9k^2}{32} + \frac{3k}{2} - \frac{3}{2} - \frac{3k^2}{16} - k + 1}{\frac{3k}{4} + \frac{3k}{4}} = -1$$ 1M

$$\frac{1}{2}\left(\frac{3k^2}{16} + k - 1\right)^2 = \frac{9k^2}{8}$$

$$\frac{3k^2}{16} + k - 1 = \pm\frac{3k}{2}$$

If $\frac{3k^2}{16} + k - 1 = \frac{3k}{2}$,

$\frac{3k^2}{16} - \frac{k}{2} - 1 = 0$

$k = 4$ or $-\frac{4}{3}$ (rejected) 1A

If $\frac{3k^2}{16} + k - 1 = -\frac{3k}{2}$,

$\frac{3k^2}{16} + \frac{5k}{2} - 1 = 0$

$k \approx -13.7$ (rejected) or -0.389 (rejected)

Coordinates of T are $(3, 9)$, and the circumcentre of $\triangle OST$ is at the midpoint of OT.

Required coordinates are $\left(\frac{3}{2}, \frac{9}{2}\right)$. 1A

8. (a) $B(-3,\ 4)$ 1A

Axis of symmetry of P is $x = \frac{6a}{2a}$, i.e., $x = 3$. 1A

The coordinates of C are $(9,\ 4)$. 1A

(b) $f(3) = 9a - 18a + 9a + b = b$ 1M

The coordinates of vertex of P are $(3,\ b)$.

$$b - (-4) = 5$$

$$b = 1$$ 1A

P passes through B.

$$a(-3)^2 - 6(-3) + (9a + 1) = 4$$

$$36a + 1 = 4$$

$$a = \frac{1}{12}$$ 1A

(c) (i) Area of $ABDC$ is the greatest when AD is a diameter, i.e., $\angle ABD = 90°$. 1M

By symmetry, the coordinates of D are $(3,\ k)$, where k is a constant.

$$\frac{k-4}{3+3} \times \frac{4+4}{-3-3} = -1$$ 1M

$$k = \frac{17}{2}$$ 1A

The coordinates of D are $\left(3,\ \frac{17}{2}\right)$.

(ii) $AB = \sqrt{(3+3)^2 + (4+4)^2} = 10$

$BD = \sqrt{(3+3)^2 + \left(\frac{17}{2} - 4\right)^2} = \frac{15}{2}$ 1A

Let the radius of the inscribed circle be r.

$$\frac{\frac{15}{2} - r}{r} = \frac{\left(\frac{15}{2}\right)}{10}$$ 1M

$$r = \frac{30}{7}$$

Area of the circle $= \pi\left(\frac{30}{7}\right)^2$

$< 25\pi$

The claim is agreed. 1A

9. (a) $g(x) = \log_3 \frac{x}{4} - 3$ 1A+1A

(b) $g(x) = \log_3 \frac{x}{4} - 3$

$= \log_3 \left(\frac{x}{4} \div 3^3 \right)$

$= \log_3 \frac{x}{108}$ 1A

The graph of $y = g(x)$ can be obtained by enlarging the graph of $y = \log_3 x$ along the x-axis to 108 times the original. 1A

The claim is agreed. 1A

10. (a) $f(3) = \frac{1}{k+2}[3^2 + (2k-2)3 - 5k - 1]$

$= \frac{1}{k+2}(k+2)$

$= 1$

The graph of $y = f(x)$ passes through A. 1

(b) (i) $g(x) = f(-x) - 2$ 1M

$= \frac{1}{k+2}[x^2 - (2k-2)x - 7k - 5]$

$= \frac{1}{k+2}[(x - 2(k-1)x + (k-1)^2) - k^2 - 5k - 6]$ 1M

$= \frac{1}{k+2}[(x - (k-1)^2) - k^2 - 5k - 6]$

$= \frac{1}{k+2}[x - (k-1)]^2 - \frac{(k+2)(k+3)}{k+2}$

$= \frac{1}{k+2}[x - (k-1)]^2 - k - 3$

The coordinates of M are $(k-1,\ -k-3)$. 1A

(ii) AN is a diameter of the circumcircle of $\triangle ANM$.

So, $\angle AMN = 90°$. 1M

$\frac{(-k-3)+9}{(k-1)-1} \times \frac{1-(-k-3)}{3-(k-1)} = -1$ 1M

$-k^2 + 2k + 24 = k^2 - 6k + 8$

$-2k^2 + 8k + 16 = 0$

$k = 2 + 2\sqrt{3}$ or $2 - 2\sqrt{3}$ (rejected) 1A

(iii) The coordinates of P are $(-3,\ -1)$. 1A

The coordinates of Q are $\left(1 + 2\sqrt{3},\ -5 - 2\sqrt{3}\right)$.

Circumcentre S lies on AN. So, S is the midpoint of AN.

The coordinates of S are $(2,\ -4)$. 1A

Slope of $PS = \frac{-1+4}{-3-2} = \frac{-3}{5}$ 1M

Slope of $PQ = \frac{-5 - 2\sqrt{3} + 1}{1 + 2\sqrt{3} + 3} = -1 \neq \frac{-3}{5}$

P, Q, S are not collinear.

The claim is disagreed. 1A