Dexter Wong & His Mathematics Team

暑期課程　2022 – 2023

數學　延伸部分

S5 – S6 M2 積分　功課　第 4 套

名字：_____　　分校：_____

上課日：日 / 一 / 二 / 三 / 四 / 五 / 六

考生須知

1. 本試卷**各題均須作答**，答案須寫在本試題答題簿中預留的空位內。
2. 除特別指明外，須詳細列出所有算式。
3. 除特別指明外，數值答案須用真確值表示。
4. 本試卷的附圖不一定依比例繪成。

建議題解

派發於暑期課程
S5 – S6 M2 積分
第 1 期 – 第 4 堂

1. $\int x \cdot 7^x \, dx = \int xe^{x \ln 7} \, dx$ 1M

$= \dfrac{xe^{x \ln 7}}{\ln 7} - \dfrac{1}{\ln 7} \int e^{x \ln 7} \, dx$ 1M

$= \dfrac{x \cdot 7^x}{\ln 7} - \dfrac{7^x}{(\ln 7)^2} + \text{常數}$ 1A

2. $\int \sec^2 x \ln(\sin x) \, dx = \tan x \ln(\sin x) - \int \tan x \cdot \dfrac{\cos x}{\sin x} \, dx$ 1M

$= \tan x \ln(\sin x) - \int dx$

$= \tan x \ln(\sin x) - x + \text{常數}$ 1A

3. $\int e^{\sqrt{x}} \, dx = \int \sqrt{x} \cdot \dfrac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$

設 $u = \sqrt{x}$。則 $du = \dfrac{1}{2\sqrt{x}} \, dx$。

$\int e^{\sqrt{x}} \, dx = 2 \int u \cdot e^u \, du$

$= 2ue^u - 2 \int e^u \, du$ 1M

$= 2ue^u - 2e^u + \text{常數}$

$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + \text{常數}$ 1A

4. $\int x^2 \cos(3x - 4) \, dx = \dfrac{x^2 \sin(3x - 4)}{3} - \dfrac{2}{3} \int x \sin(3x - 4) \, dx$ 1M

$= \dfrac{x^2 \sin(3x - 4)}{3} + \dfrac{2x \cos(3x - 4)}{9} - \dfrac{2}{9} \int \cos(3x - 4) \, dx$ 1M

$= \dfrac{x^2 \sin(3x - 4)}{3} + \dfrac{2x \cos(3x - 4)}{9} - \dfrac{2 \sin(3x - 4)}{27} + \text{常數}$ 1A

5. $\int x^5 (\ln x)^2 \, dx = \dfrac{x^6 (\ln x)^2}{6} - \dfrac{1}{3} \int x^5 \ln x \, dx$ 1M

$= \dfrac{x^6 (\ln x)^2}{6} - \dfrac{x^6 \ln x}{18} + \dfrac{1}{18} \int x^6 5 \, dx$ 1M

$= \dfrac{x^6 (\ln x)^2}{6} - \dfrac{x^6 \ln x}{18} + \dfrac{x^6}{108} + \text{常數}$ 1A

6. $\int (x^2 + e^x)e^x \, dx = \int x^2 e^x \, dx + \int e^{2x} \, dx$

$= x^2 e^x - 2 \int xe^x \, dx + \int e^{2x} \, dx$ 1M

$= x^2 e^x - 2xe^x + 2 \int e^x \, dx + \int e^{2x} \, dx$ 1M

$= x^2 e^x - 2xe^x + 2e^x + \dfrac{1}{2} e^{2x} + \text{常數}$ 1A

7. $\displaystyle\int e^{-3x}\cos 4x\,\mathrm{d}x = -\frac{e^{-3x}\cos 4x}{3} - \frac{4}{3}\int e^{-3x}\sin 4x\,\mathrm{d}x$ 1M

$\displaystyle= -\frac{e^{-3x}\cos 4x}{3} + \frac{4e^{-3x}\sin 4x}{9} - \frac{16}{9}\int e^{-3x}\cos 4x\,\mathrm{d}x$ 1M

$\displaystyle\frac{25}{9}\int e^{-3x}\cos 4x\,\mathrm{d}x = -\frac{e^{-3x}\cos 4x}{3} + \frac{4e^{-3x}\sin 4x}{9} + \text{常數}$ 1M

$\displaystyle\int e^{-3x}\cos 4x\,\mathrm{d}x = -\frac{3e^{-3x}\cos 4x}{25} + \frac{4e^{-3x}\sin 4x}{25} + \text{常數}$ 1A

8. $\displaystyle\int x^2\cos(\ln x)\,\mathrm{d}x = \frac{x^3\cos(\ln x)}{3} + \frac{1}{3}\int x^2\sin(\ln x)\,\mathrm{d}x$ 1M

$\displaystyle= \frac{x^3\cos(\ln x)}{3} + \frac{x^3\sin(\ln x)}{9} - \frac{1}{9}\int x^2\cos(\ln x)\,\mathrm{d}x$ 1M

$\displaystyle\frac{10}{9}\int x^2\cos(\ln x)\,\mathrm{d}x = \frac{x^3\cos(\ln x)}{3} + \frac{x^3\sin(\ln x)}{9} + \text{常數}$ 1M

$\displaystyle\int x^2\cos(\ln x)\,\mathrm{d}x = \frac{3x^3\cos(\ln x)}{10} + \frac{x^3\sin(\ln x)}{10} + \text{常數}$ 1A

9. 設 $u = \sqrt{x}$。則 $\mathrm{d}u = \dfrac{1}{2\sqrt{x}}\,\mathrm{d}x$。

$\displaystyle\int \sin\sqrt{x}\,\mathrm{d}x = 2\int u\sin u\,\mathrm{d}x$ 1M

$\displaystyle= -2u\cos u + 2\int \cos u\,\mathrm{d}u$ 1M

$= -2u\cos u + 2\sin u + \text{常數}$ 1A

$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + \text{常數}$ 1A

10. $\displaystyle\int x^2\cos 3x\,\mathrm{d}x = \frac{x^2\sin 3x}{3} - \frac{2}{3}\int x\sin 3x\,\mathrm{d}x$ 1M+1A

$\displaystyle= \frac{x^2\sin 3x}{3} + \frac{2x\cos 3x}{9} - \frac{2}{9}\int \cos 3x\,\mathrm{d}x$ 1M

$\displaystyle= \frac{x^2\sin 3x}{3} + \frac{2x\cos 3x}{9} - \frac{2\sin 3x}{27} + \text{常數}$ 1A

11. (a) 設 $u = 1 + x^2$。則 $\mathrm{d}u = 2x\,\mathrm{d}x$。

$\displaystyle\int \frac{x^3}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{u-1}{u}\,\mathrm{d}u$ 1M

$\displaystyle= \frac{1}{2}\int (1 - u^{-1})\,\mathrm{d}u$

$\displaystyle= \frac{u}{2} - \frac{1}{2}\ln|u| + \text{常數}$

$\displaystyle= \frac{x^2}{2} - \frac{1}{2}\ln\left(1+x^2\right) + \text{常數}$ 1A

(b) $\displaystyle\int x^2\tan^{-1}x\,\mathrm{d}x = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,\mathrm{d}x$ 1M+1A

$\displaystyle= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln\left(1+x^2\right) + \text{常數}$ 1A

12. (a) 設 $2x = \tan\theta$。則 $2\,dx = \sec^2\theta\,d\theta$。 1M

$$\int \frac{1}{1+4x^2}\,dx = \frac{1}{2}\int \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta \quad \text{1M}$$

$$= \frac{\theta}{2} + \text{常數}$$

$$= \frac{1}{2}\tan^{-1} 2x + \text{常數} \quad \text{1A}$$

(b) $\displaystyle\int \ln(1+4x^2)\,dx = x\ln(1+4x^2) - \int x\cdot\frac{8x}{1+4x^2}\,dx$ 1M+1M

$$= x\ln(1+4x^2) - 2\int\left(1 - \frac{1}{1+4x^2}\right)dx \quad \text{1M}$$

$$= x\ln(1+4x^2) - 2x + \tan^{-1} 2x + \text{常數} \quad \text{1A}$$

13. $\displaystyle\int e^{2x}(\sin x + \cos x)^2\,dx = \int e^{2x}(1 + 2\sin x\cos x)\,dx$ 1M

$$= \frac{e^{2x}}{2} + \int e^{2x}\sin 2x\,dx \quad \text{1M}$$

考慮積分 $\displaystyle\int e^{2x}\sin 2x\,dx$，

$$\int e^{2x}\sin 2x\,dx = \frac{e^{2x}}{2}\sin 2x - \int \frac{e^{2x}}{2}(2\cos 2x)\,dx \quad \text{1M}$$

$$= \frac{e^{2x}\sin 2x}{2} - \frac{e^{2x}}{2}\cos 2x + \int \frac{e^{2x}}{2}(-2\sin 2x)\,dx \quad \text{1M}$$

$$2\int e^{2x}\sin 2x\,dx = \frac{e^{2x}\sin 2x}{2} - \frac{e^{2x}\cos 2x}{2} + \text{常數} \quad \text{1M}$$

$$\int e^{2x}\sin 2x\,dx = \frac{e^{2x}}{4}(\sin 2x - \cos 2x) + \text{常數} \quad \text{1A}$$

故此，$\displaystyle\int e^{2x}(\sin x + \cos x)^2\,dx = \frac{e^{2x}}{2} + \frac{e^{2x}}{4}(\sin 2x - \cos 2x) + \text{常數}$ 1A

14. (a) 利用長除法，

$$\frac{x^7}{x^2+1} = \frac{(x^5 - x^3 + x)(x^2+1) - x}{x^2+1} \quad \text{1M}$$

$$= x^5 - x^3 + x - \frac{x}{x^2+1} \quad \text{1A}$$

(b) $\displaystyle\int x^5\ln(x^2+1)\,dx = \frac{x^6\ln(x^2+1)}{6} - \frac{1}{6}\int x^6\times\frac{2x}{x^2+1}\,dx$ 1M

$$= \frac{x^6\ln(x^2+1)}{6} - \frac{1}{3}\int\left(x^5 - x^3 + x - \frac{x}{x^2+1}\right)dx \quad \text{1M}$$

$$= \frac{x^6\ln(x^2+1)}{6} - \frac{x^6}{18} + \frac{x^4}{12} - \frac{x^2}{6} + \frac{1}{6}\int\frac{d(x^2+1)}{x^2+1} \quad \text{1M}$$

$$= \frac{x^6\ln(x^2+1)}{6} + \frac{\ln(x^2+1)}{6} - \frac{x^6}{18} + \frac{x^4}{12} - \frac{x^2}{6} + \text{常數} \quad \text{1A}$$

15. (a) 設 $u = x^2 + 3x + 1$。則 $du = (2x + 3)\,dx$。 1M

$$\int (2x + 3)\sqrt{x^2 + 3x + 1}\,dx = \int \sqrt{u}\,du \quad \text{1A}$$

$$= \frac{2u^{\frac{3}{2}}}{3} + \text{常數}$$

$$= \frac{2(x^2 + 3x + 1)^{\frac{3}{2}}}{3} + \text{常數} \quad \text{1A}$$

(b) $\int \sin^4 x\,dx = \frac{1}{4}\int (1 - \cos 2x)^2\,dx$ 1M

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int \left[1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right]dx \quad \text{1M}$$

$$= \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x)\,dx$$

$$= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + \text{常數} \quad \text{1A}$$

(c) 設 $u = x^2$。則 $du = 2x\,dx$。 1M

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int ue^u\,du$$

$$= \frac{ue^u}{2} - \frac{1}{2}\int e^u\,du \quad \text{1M}$$

$$= \frac{ue^u}{2} - \frac{e^u}{2} + \text{常數} \quad \text{1A}$$

$$= \frac{x^2e^{x^2}}{2} - \frac{e^{x^2}}{2} + \text{常數} \quad \text{1A}$$

16. (a) 設 $2x = \tan\theta$。則 $2\,dx = \sec^2\theta\,d\theta$。 1M

$$\int \frac{dx}{1+4x^2} = \frac{1}{2}\int \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta \quad \text{1A}$$

$$= \frac{\theta}{2} + \text{常數}$$

$$= \frac{1}{2}\tan^{-1} 2x + \text{常數} \quad \text{1A}$$

(b) $\displaystyle\int \frac{x^2}{1+4x^2}\,dx = \frac{1}{4}\int \left(1 - \frac{1}{1+4x^2}\right) dx$ 1M+1A

$$= \frac{x}{4} - \frac{1}{8}\tan^{-1} 2x + \text{常數} \quad \text{1A}$$

(c) 設 $x = \cos^2\theta$。則 $dx = -2\sin\theta\cos\theta\,d\theta$。 1M

$$\int \sin 2\theta \ln\left(1+4\cos^4\theta\right) d\theta$$

$$= \int 2\sin\theta\cos\theta \ln\left(1+4\cos^4\theta\right) d\theta$$

$$= -\int \ln\left(1+4x^2\right) dx \quad \text{1A}$$

$$= -x\ln\left(1+4x^2\right) + \int x \times \frac{(8x)}{1+4x^2}\,dx \quad \text{1M}$$

$$= -x\ln\left(1+4x^2\right) + 8\times\left(\frac{x}{4} - \frac{1}{8}\tan^{-1} 2x\right) + \text{常數} \quad \text{1M}$$

$$= -x\ln\left(1+4x^2\right) + 2x - \tan^{-1} 2x + \text{常數}$$

$$= -\cos^2\theta\ln\left(1+4\cos^4\theta\right) + 2\cos^2\theta - \tan^{-1}(2\cos^2\theta) + \text{常數} \quad \text{1A}$$