

Dexter Wong & His Mathematics Team

暑期課程 2022 – 2023

數學 延伸部分

S5 – S6 M2 積分 功課 第 3 套

名字：_____

分校：_____

上課日：日 / 一 / 二 / 三 / 四 / 五 / 六

考生須知

1. 本試卷各題均須作答，答案須寫在本試題答題簿中預留的空位內。
2. 除特別指明外，須詳細列出所有算式。
3. 除特別指明外，數值答案須用真確值表示。
4. 本試卷的附圖不一定依比例繪成。

建議題解



派發於暑期課程
S5 – S6 M2 積分
第 1 期 – 第 3 堂

1. 設 $x = e \sin \theta$ 。則 $dx = e \cos \theta d\theta$ 。 1M

$$\int x^2 \sqrt{e^2 - x^2} dx = \int (e \sin \theta)^2 \sqrt{e^2 - e^2 \sin^2 \theta} e \cos \theta d\theta$$

$$= e^4 \int \sin^2 \theta \cos^2 \theta d\theta$$

1A

$$= \frac{e^4}{4} \int (2 \sin \theta \cos \theta)^2 d\theta$$

$$= \frac{e^4}{4} \int \sin^2 2\theta d\theta$$

1M

$$= \frac{e^4}{8} \int (1 - \cos 4\theta) d\theta$$

1M

$$= \frac{e^4 \theta}{8} - \frac{e^4 \sin 4\theta}{32} + \text{常數}$$

$$= \frac{e^4 \theta}{8} - \frac{e^4}{16} \sin 2\theta \cos 2\theta + \text{常數}$$

$$= \frac{e^4 \theta}{8} - \frac{e^4}{8} \sin \theta \cos \theta (1 - 2 \sin^2 \theta) + \text{常數}$$

$$= \frac{e^4 \theta}{8} - \frac{e^4}{8} \left(\frac{x}{e} \right) \left(\frac{\sqrt{e^2 - x^2}}{e} \right) \left[1 - 2 \left(\frac{x}{e} \right)^2 \right] + \text{常數}$$

$$= \frac{e^4}{8} \sin^{-1} \frac{x}{e} - \frac{1}{8} x (e^2 - 2x^2) \sqrt{e^2 - x^2} + \text{常數}$$

1A

2. (a) $4x - x^2 = 4 - (x - 2)^2$ 1A

- (b) 設 $x - 2 = 2 \sin \theta$ 。則 $dx = 2 \cos \theta d\theta$ 。 1M

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{2 \cos \theta}{\sqrt{4 - 4 \sin^2 \theta}} d\theta$$

$$= \int \frac{\cos \theta}{\cos \theta} d\theta$$

$$= \int d\theta$$

1A

$$= \theta + \text{常數}$$

$$= \sin^{-1} \frac{x - 2}{2} + \text{常數}$$

1A

3. $x^2 + 8x + 17 = (x + 4)^2 + 1$

設 $x + 4 = \tan \theta$ 。則 $dx = \sec^2 \theta d\theta$ 。

1M

$$\begin{aligned}\int \frac{x}{x^2 + 8x + 17} dx &= \int \frac{(\tan \theta - 4) \sec^2 \theta}{\tan^2 \theta + 1} d\theta \\ &= \int (\tan \theta - 4) d\theta\end{aligned}$$

1A

設 $u = \cos \theta$ 。則 $du = -\sin \theta d\theta$ 。

1M

$$\begin{aligned}\int \frac{x}{x^2 + 8x + 17} dx &= -\int \frac{du}{u} - \int 4 d\theta \\ &= -\ln |\cos \theta| - 4\theta + \text{常數} \\ &= -\ln \left| \frac{1}{\sqrt{x^2 + 8x + 17}} \right| - 4 \tan^{-1}(x + 4) + \text{常數} \\ &= \ln \sqrt{x^2 + 8x + 17} - 4 \tan^{-1}(x + 4) + \text{常數}\end{aligned}$$

1A

4. 設 $\ln x = \sin \theta$ 。則 $\frac{1}{x} dx = \cos \theta d\theta$ 。

1M

$$\begin{aligned}\int \frac{\sqrt{1 - (\ln x)^2}}{x} dx &= \int \sqrt{1 - \sin^2 \theta} \cos \theta d\theta \\ &= \int \cos^2 \theta d\theta\end{aligned}$$

1A

$$= \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$

1M

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{常數}$$

$$= \frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} + \text{常數}$$

$$= \frac{1}{2} \sin^{-1}(\ln x) + \frac{1}{2} (\ln x) \sqrt{1 - (\ln x)^2} + \text{常數}$$

1A

5. 設 $e^x = \tan \theta$ 。則 $e^x dx = \sec^2 \theta d\theta$ 。

1M

$$\begin{aligned}\int \frac{e^x dx}{(1 + e^{2x})^2} &= \int \frac{\sec^2 \theta}{(1 + \tan^2 \theta)^2} d\theta \\ &= \int \cos^2 \theta d\theta\end{aligned}$$

1A

$$= \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$

1M

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{常數}$$

$$= \frac{\theta}{2} + \frac{1}{2} \sin \theta \cos \theta + \text{常數}$$

$$= \frac{1}{2} \tan^{-1} e^x + \frac{1}{2} \left(\frac{e^x}{\sqrt{1 + e^{2x}}} \right) \left(\frac{1}{\sqrt{1 + e^{2x}}} \right) + \text{常數}$$

$$= \frac{1}{2} \tan^{-1} e^x + \frac{e^x}{2(1 + e^{2x})} + \text{常數}$$

1A

6. 設 $x = \sqrt{3} \sin \theta$ 。則 $dx = \sqrt{3} \cos \theta d\theta$ 。 1M

$$\begin{aligned} \int \sqrt{1 - \frac{x^2}{3}} dx &= \frac{1}{\sqrt{3}} \int \sqrt{3 - 3 \sin^2 \theta} \cdot \sqrt{3} \cos \theta d\theta \\ &= \sqrt{3} \int \cos^2 \theta d\theta \end{aligned} \quad 1A$$

$$= \frac{\sqrt{3}}{2} \int (1 + \cos 2\theta) d\theta \quad 1M$$

$$= \frac{\sqrt{3}\theta}{2} + \frac{\sqrt{3}}{4} \sin 2\theta + \text{常數}$$

$$= \frac{\sqrt{3}\theta}{2} + \frac{\sqrt{3}}{2} \sin \theta \cos \theta + \text{常數}$$

$$= \frac{\sqrt{3}}{2} \sin^{-1} \frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{2} \left(\frac{x}{\sqrt{3}} \right) \left(\frac{\sqrt{3-x^2}}{\sqrt{3}} \right) + \text{常數}$$

$$= \frac{\sqrt{3}}{2} \sin^{-1} \frac{x}{\sqrt{3}} + \frac{x\sqrt{3(3-x^2)}}{6} + \text{常數} \quad 1A$$

7. (a) $x^2 + 2x \sin \frac{5\pi}{12} + 1 = x^2 + 2x \sin \frac{5\pi}{12} + \sin^2 \frac{5\pi}{12} + \cos^2 \frac{5\pi}{12}$
 $= \left(x + \sin \frac{5\pi}{12} \right)^2 + \cos^2 \frac{5\pi}{12} \quad 1A$

(b) 設 $x + \sin \frac{5\pi}{12} = \cos \frac{5\pi}{12} \tan \theta$ 。則 $dx = \cos \frac{5\pi}{12} \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{\cos \frac{5\pi}{12}}{x^2 + 2x \sin \frac{5\pi}{12} + 1} dx &= \int \frac{\cos \frac{5\pi}{12}}{\cos^2 \frac{5\pi}{12} \tan^2 \theta + \cos^2 \frac{5\pi}{12}} \cdot \cos \frac{5\pi}{12} \sec^2 \theta d\theta \\ &= \int d\theta \end{aligned} \quad 1A$$

$$= \theta + \text{常數}$$

$$= \tan^{-1} \left(x \csc \frac{5\pi}{12} + \cot \frac{5\pi}{12} \right) + \text{常數}$$

$$= \tan^{-1} [(\sqrt{6} - \sqrt{2})x + 2 - \sqrt{3}] + \text{常數} \quad 1A$$

$$8. \quad (a) \quad x^2 + 2x \cos \frac{7\pi}{12} + 1 = x^2 + 2x \cos \frac{7\pi}{12} + \sin^2 \frac{7\pi}{12} + \cos^2 \frac{7\pi}{12} \\ = \left(x + \cos \frac{7\pi}{12}\right)^2 + \sin^2 \frac{7\pi}{12} \quad 1A$$

$$(b) \quad \text{設 } x + \cos \frac{7\pi}{12} = \sin \frac{7\pi}{12} \tan \theta \circ \text{ 則 } dx = \sin \frac{7\pi}{12} \sec^2 \theta d\theta \circ \quad 1M$$

$$\int \frac{1}{x^2 + 2x \cos \frac{7\pi}{12} + 1} dx = \int \frac{\sin \frac{7\pi}{12} \sec^2 \theta}{\sin^2 \frac{7\pi}{12} \tan^2 \theta + \sin^2 \frac{7\pi}{12}} d\theta \\ = \csc \frac{7\pi}{12} \int d\theta \quad 1A$$

$$= \csc \frac{7\pi}{12} \theta + \text{常數} \\ = \csc \frac{7\pi}{12} \tan^{-1} \left(x \csc \frac{7\pi}{12} + \cot \frac{7\pi}{12} \right) + \text{常數} \\ = (\sqrt{6} - \sqrt{2}) \tan^{-1} [(\sqrt{6} - \sqrt{2})x + \sqrt{3} - 2] + \text{常數} \quad 1A$$

$$9. \quad (a) \quad x^2 + 2x + 5 = (x + 1)^2 + 4 \quad 1A$$

$$(b) \quad \text{設 } x + 1 = 2 \tan \theta \circ \text{ 則 } dx = 2 \sec^2 \theta d\theta \circ \quad 1M$$

$$\int \frac{dx}{(x + 1)^2(x^2 + 2x + 5)} = \int \frac{2 \sec^2 \theta}{(2 \tan \theta)^2(4 \tan^2 \theta + 4)} d\theta \quad 1M+1A$$

$$= \frac{1}{8} \int \cot^2 \theta d\theta \\ = \frac{1}{8} \int (\csc^2 \theta - 1) d\theta \quad 1M$$

$$= -\frac{1}{8} \cot \theta - \frac{\theta}{8} + \text{常數} \\ = -\frac{1}{8} \left(\frac{2}{x+1} \right) - \frac{1}{8} \tan^{-1} \frac{x+1}{2} + \text{常數} \\ = -\frac{1}{4(x+1)} - \frac{1}{8} \tan^{-1} \frac{x+1}{2} + \text{常數} \quad 1A$$

10. (a) 設 $u = \tan x$ 。則 $du = \sec^2 x \, dx$ 。 1M

$$\begin{aligned} \int \sec^4 x \tan^3 x \, dx &= \int (u^2 + 1)u^3 \, du & 1M \\ &= \int (u^5 + u^3) \, du \\ &= \frac{u^6}{6} + \frac{u^4}{4} + \text{常數} \\ &= \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + \text{常數} & 1A \end{aligned}$$

- (b) 設 $x = 5 \tan \theta$ 。則 $dx = 5 \sec^2 \theta \, d\theta$ 。 1M

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{25 + x^2}} \, dx &= \int \frac{5 \sec^2 \theta}{25 \tan^2 \theta \sqrt{25 + 25 \tan^2 \theta}} \, d\theta \\ &= \frac{1}{25} \int \frac{\sec \theta}{\tan^2 \theta} \, d\theta \\ &= \frac{1}{25} \int \frac{\cos \theta}{\sin^2 \theta} \, d\theta & 1M \end{aligned}$$

- 設 $w = \sin \theta$ 。則 $dw = \cos \theta \, d\theta$ 。 1M

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{25 + x^2}} \, dx &= \frac{1}{25} \int \frac{dw}{w^2} \\ &= -\frac{1}{25w} + \text{常數} \\ &= -\frac{\sqrt{25 + x^2}}{25x} + \text{常數} & 1A \end{aligned}$$

11. (a) 設 $x = \sin^2 \theta$ 。則 $dx = 2 \sin \theta \cos \theta \, d\theta$ 。

$$\begin{aligned} \int \frac{f(x)}{\sqrt{x(1-x)}} \, dx &= \int \frac{f(\sin^2 \theta) \cdot 2 \sin \theta \cos \theta}{\sqrt{\sin^2 \theta (1 - \sin^2 \theta)}} \, d\theta \\ &= 2 \int f(\sin^2 \theta) \, d\theta & 1 \end{aligned}$$

(b) $\int \frac{dx}{\sqrt{x(1-x)}} = 2 \int d\theta$

$$\begin{aligned} &= 2\theta + \text{常數} \\ &= 2 \sin^{-1} \sqrt{x} + \text{常數} & 1A \end{aligned}$$

$$\begin{aligned} \int \sqrt{\frac{x}{1-x}} \, dx &= \int \frac{x}{\sqrt{x(1-x)}} \, dx \\ &= 2 \int \sin^2 \theta \, d\theta & 1A \\ &= \int (1 - \cos 2\theta) \, d\theta \\ &= \theta - \frac{1}{2} \sin 2\theta + \text{常數} & 1A \\ &= \sin^{-1} \sqrt{x} - \sqrt{x(1-x)} + \text{常數} & 1A \end{aligned}$$

12. (a) 設 $x = a \tan \theta$ 。則 $dx = a \sec^2 \theta d\theta$ 。

$$\begin{aligned} \int \frac{1}{x^2 + a^2} dx &= \int \frac{a \sec^2 \theta}{a^2 \tan^2 \theta + a^2} d\theta & 1M+1A \\ &= \frac{1}{a} \theta + \text{常數} \\ &= \frac{1}{a} \tan^{-1} \frac{x}{a} + \text{常數} & 1 \end{aligned}$$

(b) (i) $t = \tan \frac{\theta}{2}$

$$\begin{aligned} 1 &= \sec^2 \frac{\theta}{2} \times \frac{1}{2} \frac{d\theta}{dt} & 1A \\ \frac{d\theta}{dt} &= \frac{2}{1 + \tan^2 \frac{\theta}{2}} \\ &= \frac{2}{1 + t^2} & 1 \end{aligned}$$

(ii) 設 $t = \tan \frac{\theta}{2}$ 。則 $d\theta = \frac{2}{1 + t^2} dt$ 。

$$\cos \theta = 2 \cos^2 \frac{\theta}{2} - 1 = \frac{2}{1 + t^2} - 1 = \frac{1 - t^2}{1 + t^2}。$$

$$\begin{aligned} \int \frac{1}{2 + \cos \theta} d\theta &= \int \frac{\frac{2}{1+t^2}}{2 + \frac{1-t^2}{1+t^2}} dt & 1M \\ &= \int \frac{2}{2 + 2t^2 + 1 - t^2} dt \\ &= 2 \int \frac{1}{t^2 + 3} dt \\ &= \frac{2}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + \text{常數} & 1M \\ &= \frac{2}{\sqrt{3}} \tan^{-1} \frac{\tan \frac{\theta}{2}}{\sqrt{3}} + \text{常數} & 1A \end{aligned}$$

$$13. \quad (a) \quad (i) \quad \frac{A}{u^2} + \frac{B}{u-1} + \frac{C}{u+1} = \frac{A(u+1)(u-1) + Bu^2(u+1) + Cu^2(u-1)}{u^2(u-1)(u+1)}$$

故此，可得 $1 \equiv A(u-1)(u+1) + Bu^2(u+1) + Cu^2(u-1)$ 。

1M

代 $u = 0$ ，則 $A = -1$ ；代 $u = 1$ ，則 $B = \frac{1}{2}$ ；代 $u = -1$ ，則 $C = -\frac{1}{2}$ 。

1A+1A

$$(ii) \quad \int \frac{du}{u^2(u-1)(u+1)} du = \int \left(-\frac{1}{u^2} + \frac{1}{2(u-1)} - \frac{1}{2(u+1)} \right) du$$

$$= \frac{1}{u} + \frac{1}{2} \ln|u-1| - \frac{1}{2} \ln|u+1| + \text{常數}$$

$$= \frac{1}{u} + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + \text{常數}$$

1M+1A

(備註) 若寫出了 $\boxed{+C}$ ，則 pp -1。

C 在題目中已使用了，不能用作表示常數。

(b) (i) 設 $x = 3 \tan \theta$ 。則 $dx = 3 \sec^2 \theta d\theta$ 。

$$\int \frac{\sqrt{x^2+9}}{x} dx = \int \frac{\sqrt{9 \tan^2 \theta + 9}}{3 \tan \theta} \times 3 \sec^2 \theta d\theta$$

$$= 3 \int \frac{\sec^3 \theta}{\tan \theta} d\theta$$

$$= 3 \int \frac{d\theta}{\sin \theta \cos^2 \theta}$$

1A

1

(ii) 設 $u = \cos \theta$ 。則 $du = -\sin \theta d\theta$ 。

$$\int \frac{\sqrt{x^2+9}}{x} dx = 3 \int \frac{d\theta}{\sin \theta \cos^2 \theta}$$

$$= 3 \int \frac{-\sin \theta}{(\cos \theta + 1)(\cos \theta - 1) \cos^2 \theta} d\theta$$

$$= 3 \int \frac{du}{u^2(u+1)(u-1)}$$

$$= \frac{3}{u} + \frac{3}{2} \ln \left| \frac{u-1}{u+1} \right| + \text{常數}$$

$$= 3 \div \frac{3}{\sqrt{x^2+9}} + \frac{3}{2} \ln \left| \frac{\frac{3}{\sqrt{x^2+9}} - 1}{\frac{3}{\sqrt{x^2+9}} + 1} \right| + \text{常數}$$

$$= \sqrt{x^2+9} + \frac{3}{2} \ln \left| \frac{3 - \sqrt{x^2+9}}{3 + \sqrt{x^2+9}} \right| + \text{常數}$$

1M

1M

1A