Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Extended Part

S5 – S6 M2 Integration Assignment $\boxed{\text{Set 2}}$

Name: ____________________ Telephone: __________

Student number: ____________________ Centre: __________

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings must be clearly shown.
3. Unless otherwise specified, numerical answers must be exact.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution

Distributed in summer course
S5 – S6 M2 Integration
Phase 1 – Lesson 2

1. (a) Let $u = x + 2$. Then $\mathrm{d}u = \mathrm{d}x$. 1M

$$\int \frac{x-1}{x+2}\,\mathrm{d}x = \int \frac{u-3}{u}\,\mathrm{d}u$$

$$= \int \left(1 - \frac{3}{u}\right)\mathrm{d}u$$

$$= u - 3\ln|u| + \text{constant}$$

$$= x - 3\ln|x+2| + \text{constant} \qquad \text{1A}$$

(b) Let $u = \sqrt{x}$. Then $\mathrm{d}u = \dfrac{1}{2\sqrt{x}}\,\mathrm{d}x$. 1M

$$\int \frac{\sqrt{x}-1}{x+2\sqrt{x}}\,\mathrm{d}x = 2\int \left(\frac{\sqrt{x}-1}{\sqrt{x}+2} \cdot \frac{1}{2\sqrt{x}}\right)\mathrm{d}x$$

$$= 2\int \frac{u-1}{u+2}\,\mathrm{d}u \qquad \text{1A}$$

$$= 2u - 6\ln|u+2| + \text{constant} \qquad \text{1A}$$

$$= 2\sqrt{x} - 6\ln\left(\sqrt{x}+2\right) + \text{constant} \qquad \text{1A}$$

2. (a) Let $u = x - 1$. Then $\mathrm{d}u = \mathrm{d}x$.

$$\int x(x-1)^n\,\mathrm{d}x = \int (u+1)u^n\,\mathrm{d}u \qquad \text{1M}$$

$$= \int (u^{n+1} + u^n)\,\mathrm{d}u \qquad \text{1A}$$

$$= \frac{(x-1)^{n+2}}{n+2} + \frac{(x-1)^{n+1}}{n+1} + \text{constant} \qquad \text{1A}$$

(b) Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$.

$$\int \sin 2\theta(\sin\theta - 1)^{2014}\,\mathrm{d}\theta = 2\int \sin\theta\cos\theta(\sin\theta-1)^{2014}\,\mathrm{d}\theta$$

$$= 2\int x(x-1)^{2014}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{(\sin\theta-1)^{2016}}{1008} + \frac{2(\sin\theta-1)^{2015}}{2015} + \text{constant} \qquad \text{1A}$$

3. (a) $$\frac{\cot^2\theta - 1}{1+\cot^2\theta} = \frac{\cos^2\theta - \sin^2\theta}{\sin^2\theta + \cos^2\theta} \qquad \text{1M}$$

$$= \frac{\cos 2\theta}{1} = \cos 2\theta \qquad \text{1}$$

(b) $$\int \frac{\sin\theta\cos\theta(1+\cot^2\theta)}{\cot^2\theta - 1}\,\mathrm{d}\theta = \int \frac{\sin\theta\cos\theta}{\cos 2\theta}\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{2}\int \frac{\sin 2\theta}{\cos 2\theta}\,\mathrm{d}\theta$$

$$= -\frac{1}{4}\int \frac{\mathrm{d}u}{u} \qquad (\text{where } u = \cos 2\theta) \qquad \text{1M}$$

$$= -\frac{1}{4}\ln|\cos 2\theta| + \text{constant} \qquad \text{1A}$$

4. (a) Let $u = x^2 - 8x + 32$. Then $\mathrm{d}u = (2x - 8)\,\mathrm{d}x$. 1M

$$\int \frac{x-4}{x^2-8x+32}\,\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}u}{u}$$ 1M

$$= \frac{\ln|u|}{2} + \text{constant}$$

$$= \frac{\ln|x^2-8x+32|}{2} + \text{constant}$$ 1A

(b) $$\int \frac{x^2\,\mathrm{d}x}{x^2-8x+32} = \int \left(1 + \frac{8x-32}{x^2-8x+32}\right)\mathrm{d}x$$ 1M

$$= x + 4\ln|x^2-8x+32| + \text{constant}$$ 1A

5. (a) Let $u = x^2 + 2x + 2$. Then $\mathrm{d}u = (2x + 2)\,\mathrm{d}x$. 1M

$$\int \frac{2x+2}{x^2+2x+2}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u}$$ 1M

$$= \ln|u| + \text{constant}$$

$$= \ln|x^2+2x+2| + \text{constant}$$ 1A

(b) $$\int \frac{x^2}{x^2+2x+2}\,\mathrm{d}x = \int \left(1 - \frac{2x+2}{x^2+2x+2}\right)\mathrm{d}x$$ 1M

$$= x - \ln|x^2+2x+2| + \text{constant}$$ 1A

6. Let $u = 1 + \frac{3}{x^2}$. Then $\mathrm{d}u = -\frac{6}{x^3}\,\mathrm{d}x$.

$$\int \frac{\sqrt{x^2+3}}{x^4}\,\mathrm{d}x = -\frac{1}{6}\int \left(\frac{\sqrt{x^2+3}}{x}\right)\left(-\frac{6}{x^3}\right)\mathrm{d}x$$

$$= -\frac{1}{6}\int \sqrt{1+\frac{3}{x^2}}\left(-\frac{6}{x^3}\right)\mathrm{d}x$$ 1M

$$= -\frac{1}{6}\int \sqrt{u}\,\mathrm{d}u$$ 1A

$$= -\frac{1}{9}u^{\frac{3}{2}} + \text{constant}$$

$$= -\frac{1}{9}\left(1+\frac{3}{x^2}\right)^{\frac{3}{2}} + \text{constant}$$ 1A

7. Let $u = 3 + \sin(\ln x)$. Then $\mathrm{d}u = \frac{\cos(\ln x)}{x}\,\mathrm{d}x$. 1M

$$\int \frac{\cos(\ln x)}{x[3+\sin(\ln x)]}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u}$$ 1A

$$= \ln|3+\sin(\ln x)| + \text{constant}$$

$$= \ln(3+\sin(\ln x)) + \text{constant}$$ 1A

8. Let $u = \sin^2 x$. Then $\mathrm{d}u = 2\sin x\cos x\,\mathrm{d}x$. 1A

$$\int \frac{\sin x\cos x}{\sqrt{9\sin^2 x + 4\cos^2 x}}\,\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{9u + 4(1-u)}}$$ 1A

$$= \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{5u+4}}$$

$$= \frac{\sqrt{5u+4}}{5} + \text{constant}$$

$$= \frac{\sqrt{5\sin^2 x + 4}}{5} + \text{constant}$$ 1A

9. (a) $$\int \sqrt{1+\cos x}\,\mathrm{d}x = \int \sqrt{1 + (2\cos^2 x - 1)}\,\mathrm{d}x$$ 1M

$$= \sqrt{2}\int \cos\frac{x}{2}\,\mathrm{d}x$$

$$= 2\sqrt{2}\sin\frac{x}{2} + \text{constant}$$ 1A

(b) Let $u = \frac{\pi}{2} - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int \sqrt{1+\sin x}\,\mathrm{d}x = -\int \sqrt{1 + \sin\left(\frac{\pi}{2} - u\right)}\,\mathrm{d}u$$ 1A

$$= -\int \sqrt{1+\cos u}\,\mathrm{d}u$$

$$= -2\sqrt{2}\sin\frac{u}{2} + \text{constant}$$ 1M

$$= -2\sqrt{2}\sin\left(\frac{\pi}{4} - \frac{x}{2}\right) + \text{constant}$$ 1A

10. (a) Let $u = x^3$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$. 1M

$$\int x^2 2^{x^3}\,\mathrm{d}x = \frac{1}{3}\int 2^u\,\mathrm{d}u$$

$$= \frac{1}{3}\int e^{u\ln 2}\,\mathrm{d}u$$

$$= \frac{e^{u\ln 2}}{3\ln 2} + \text{constant}$$ 1A

$$= \frac{2^u}{3\ln 2} + \text{constant}$$

$$= \frac{2^{x^3}}{3\ln 2} + \text{constant}$$ 1A

(b) Let $u = x^3$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$.

$$\int x^2(2^{x^3} + 3^{x^3} + 5^{x^3})\,\mathrm{d}x = \frac{1}{3}(2^u + 3^u + 5^u)\,\mathrm{d}u$$ 1M

$$= \frac{2^u}{3\ln 2} + \frac{3^u}{3\ln 3} + \frac{5^u}{3\ln 5} + \text{constant}$$ 1M

$$= \frac{2^{x^3}}{3\ln 2} + \frac{3^{x^3}}{3\ln 3} + \frac{5^{x^3}}{3\ln 5} + \text{constant}$$ 1A

11. (a) Let $u = 1 + 4\sin^2 x$. Then $\mathrm{d}u = 8\sin x\cos x\,\mathrm{d}x$. 1M

$$\int \frac{\sin 2x}{1+4\sin^2 x}\,\mathrm{d}x = \frac{1}{4}\int \frac{8\sin x\cos x}{1+4\sin^2 x}\,\mathrm{d}x$$

$$= \frac{1}{4}\int \frac{\mathrm{d}u}{u}$$ 1A

$$= \frac{1}{4}\ln\left(1+4\sin^2 x\right) + \text{constant}$$ 1A

(b) Let $u = \sin^2 x - 3\sin x + 2$. Then $\mathrm{d}u = (2\sin x\cos x - 3\cos x)\,\mathrm{d}x$. 1M

$$\int \frac{(2\sin x - 3)\cos x}{\sin^2 x - 3\sin x + 2}\,\mathrm{d}x = \int \frac{2\sin x\cos x - 3\cos x}{\sin^2 x - 3\sin x + 2}\,\mathrm{d}x$$

$$= \int \frac{\mathrm{d}u}{u}$$ 1A

$$= \ln|\sin^2 x - 3\sin x + 2| + \text{constant}$$ 1A

12. (a) Let $u = e^{3x} - 4$. Then $\mathrm{d}u = 3e^{3x}\,\mathrm{d}x$. 1M

$$\int \frac{e^{3x}}{3^{3x}-4}\,\mathrm{d}x = \frac{1}{3}\int \frac{\mathrm{d}u}{u}$$ 1A

$$= \frac{1}{3}\ln|e^{3x}-4| + \text{constant}$$ 1A

(b) Let $u = e^{2x} - 1$. Then $\mathrm{d}u = 2e^{2x}\,\mathrm{d}x$. 1M

$$\int \frac{e^x}{e^x - e^{-x}}\,\mathrm{d}x = \frac{1}{2}\int \frac{2e^{2x}}{e^{2x}-1}\,\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{\mathrm{d}u}{u}$$ 1A

$$= \frac{1}{2}\ln|e^{2x}-1| + \text{constant}$$ 1A

13. (a) $(1+ax)^7 = \sum_{r=0}^{7} C_r^7 a^r x^r$ 1M

$$\frac{\beta_1}{\beta_3} = \frac{C_1^7 a}{C_3^7 a^3} = \frac{1}{45}$$ 1M

$$\frac{7}{35a^2} = \frac{1}{45}$$

$$a^2 = 9$$

$$a = 3 \quad \text{or} \quad -3 \text{ (rejected)}$$ 1A

(b) $$\int \sum_{r=0}^{7} \beta_r x^r\,\mathrm{d}x = \int (1+3x)^7\,\mathrm{d}x$$ 1M

$$= \frac{(1+3x)^8}{24} + \text{constant}$$ 1A

14. (a) $\int \cos 4x \cos x \, dx = \frac{1}{2}\int (\cos 5x + \cos 3x) \, dx$ 1M

$= \frac{\sin 5x}{10} + \frac{\sin 3x}{6} + \text{constant}$ 1A

(b) $\frac{\sin 6x - \sin 2x}{\sin x} = \frac{2\cos 4x \sin 2x}{\sin x}$ 1M

$= \frac{2\cos 4x(2\sin x \cos x)}{\sin x}$ 1M

$= 4\cos 4x \cos x$ 1

(c) $\int \frac{\sin 6x}{\sin x} \, dx = \int \left(\frac{\sin 6x - \sin 2x}{\sin x} + \frac{\sin 2x}{\sin x}\right) dx$

$= 4\int \cos 4x \cos x \, dx + \int \frac{\sin 2x}{\sin x} \, dx$ 1M

$= 4\int \cos 4x \cos x \, dx + 2\int \cos x \, dx$ 1M

$= 4\left(\frac{\sin 5x}{10} + \frac{\sin 3x}{6}\right) + 2\sin x + \text{constant}$ 1M

$= 2\sin x + \frac{2\sin 5x}{5} + \frac{2\sin 3x}{3} + \text{constant}$ 1A

15. (a) $\int \cos^6 \frac{x}{2} \sin 2x \, dx = \int \left(\frac{1+\cos x}{2}\right)^3 (2\sin x \cos x) \, dx$ 1M

Let $u = \cos x$. Then $du = -\sin x \, dx$. 1M

$\int \cos^6 \frac{x}{2} \sin 2x \, dx = -\frac{1}{4}\int (1+u)^3 u \, du$

$= -\frac{1}{4}(u + 3u^2 + 3u^3 + u^4) \, du$ 1A

$= -\frac{1}{4}\left(\frac{\cos^2 x}{2} + \cos^3 x + \frac{3\cos^4 x}{4} + \frac{\cos^5 x}{5}\right) + \text{constant}$

$= -\frac{\cos^2 x}{8} - \frac{\cos^3 x}{4} - \frac{3\cos^4 x}{16} - \frac{\cos^5 x}{20} + \text{constant}$ 1A

(b) Let $u = 2^x$. Then $du = 2^x \ln 2 \, dx$. 1M

$\int 2^x \cos^6 2^{x-1} \sin 2^{x+1} \, dx = \frac{1}{\ln 2}\int \cos^6 \frac{u}{2} \sin 2u \, du$ 1A

$= \frac{1}{\ln 2}\left(\frac{\cos^2 u}{8} - \frac{\cos^3 u}{4} - \frac{3\cos^4 u}{16} - \frac{\cos^5 u}{20}\right) + \text{constant}$ 1A

$= -\frac{\cos^2 2^x}{8\ln 2} - \frac{\cos^3 2^x}{4\ln 2} - \frac{3\cos^4 2^x}{16\ln 2} - \frac{\cos^5 2^x}{20\ln 2} + \text{constant}$ 1A

16. (a) $Ax + B + \dfrac{C}{x-4} + \dfrac{D}{x+2}$

$= \dfrac{(Ax+B)(x-4)(x+2) + C(x+2) + D(x-4)}{(x-4)(x+2)}$

Therefore, $2x^3 - 4x^2 - 13x \equiv (Ax+B)(x-4)(x+2) + C(x+2) + D(x-4)$.

$$\begin{cases} 2(4)^3 - 4(4)^2 - 13(4) = 6C \\ 2(-2)^3 - 4(-2)^2 - 13(-2) = -6D \\ 0 = B(-4)(2) + 2C - 4D \\ 2 - 4 - 13 = (A+B)(1-4)(1+2) + 3C - 3D \end{cases}$$ 1M

Solving, we have $A = 2$, $B = 0$, $C = 2$ and $D = 1$. 1A+1A

(b) $\displaystyle\int \frac{2x^3 - 4x^2 - 13x}{x^2 - 2x - 8}\,\mathrm{d}x = \int \left(2x + \frac{2}{x-4} + \frac{1}{x+2}\right)\mathrm{d}x$ 1M

$= x^2 + 2\ln|x-4| + \ln|x+2| + \text{constant}$ 1A

(c) Let $u = e^x$. Then $\mathrm{d}u = e^x\,\mathrm{d}x$. 1M

$$\int \frac{2e^{2x} - 4e^x - 13}{8e^{-2x} + 2e^{-x} - 1}\,\mathrm{d}x = \int \frac{(2e^{3x} - 4e^{2x} - 13e^x)e^x}{-x^2 - 2e^x + 8}\,\mathrm{d}x$$

$= -\displaystyle\int \frac{2u^3 - 4u^2 - 13u}{u^2 - 2u + 8}\,\mathrm{d}u$ 1A

$= -u^2 - 2\ln|u-4| - \ln|u+2| + \text{constant}$

$= -e^{2x} - 2\ln|e^x - 4| - \ln|e^x + 2| + \text{constant}$ 1A

17. (a) Let $u = 6\sin x - 5\cos x$. Then $\mathrm{d}u = (6\cos x + 5\sin x)\,\mathrm{d}x$. 1M

$$\int \frac{5\sin x + 6\cos x}{6\sin x - 5\cos x}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u}$$

$= \ln|6\sin x - 5\cos x| + \text{constant}$ 1A

(b) (i) $af(x) + bg(x) = (5a + 6b)\sin x + (6a - 5b)g(x)$

We have $5a + 6b = 1$ and $6a - 5b = -1$. 1M

Solving, we have $a = -\dfrac{1}{61}$ and $b = \dfrac{11}{61}$. 1A

Thus, $\sin x - \cos x = -\dfrac{1}{61}f(x) + \dfrac{11}{61}g(x)$.

(ii) $\displaystyle\int \frac{\sin x - \cos x}{6\sin x - 5\cos x}\,\mathrm{d}x = -\frac{1}{61}\int \frac{5\sin x + 6\cos x}{6\sin x - 5\cos x}\,\mathrm{d}x + \frac{11}{61}\int \frac{6\sin x - 5\cos x}{6\sin x - 5\cos x}\,\mathrm{d}x$ 1M

$= -\dfrac{1}{61}\displaystyle\int \frac{5\sin x + 6\cos x}{6\sin x - 5\cos x}\,\mathrm{d}x + \frac{11}{61}\int \mathrm{d}x$ 1M

$= -\dfrac{1}{61}\ln|6\sin x - 5\cos x| + \dfrac{11x}{61} + \text{constant}$ 1A