

Dexter Wong & His Mathematics Team
Summer Course 2022 – 2023
MATHEMATICS Extended Part
S5 – S6 M2 Differentiation Assignment Set 2

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Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings must be clearly shown.
3. Unless otherwise specified, numerical answers must be exact.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution



Distributed in summer course
S5 – S6 M2 Differentiation
Phase 1 – Lesson 2

$$1. \quad (a) \quad y = \frac{(x+4)(x-3)^2}{(x+1)^3}$$

$$= \frac{(u+3)(u-4)^2}{u^3} \quad 1A$$

$$(b) \quad y = \frac{u^3 - 5u^2 - 8u + 48}{u^3}$$

$$= 1 - 5u^{-1} - 8u^{-2} + 48u^{-3}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad 1M$$

$$= [5u^{-2} + 16u^{-3} - 144u^{-4}] (1) \quad 1A$$

$$= \frac{5}{u^2} + \frac{16}{u^3} - \frac{144}{u^4}$$

$$= \frac{5}{(x+1)^2} + \frac{16}{(x+1)^3} - \frac{144}{(x+1)^4} \quad 1A$$

$$2. \quad (a) \quad y = \frac{(3x-1)^2(4x+3)}{(2x+1)^4}$$

$$= \frac{\left[3\left(\frac{u-1}{2}\right) - 1\right]^2 \left[4\left(\frac{u-1}{2}\right) + 3\right]}{u^4} \quad 1A$$

$$= \frac{(3u-5)^2(2u+1)}{4u^2} \quad 1A$$

$$(b) \quad y = \frac{18u^3 - 51u^2 + 20u + 25}{4u^4}$$

$$= \frac{9}{2}u^{-1} - \frac{51}{4}u^{-2} + 5u^{-3} + \frac{25}{4}u^{-4}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad 1M$$

$$= \left(-\frac{9}{2u^2} + \frac{51}{2u^3} - \frac{15}{u^4} - \frac{25}{u^5}\right) (2) \quad 1A$$

$$= -\frac{9}{(2x+1)^2} + \frac{51}{(2x+1)^3} - \frac{30}{(2x+1)^4} - \frac{50}{(2x+1)^5} \quad 1$$

$$\begin{aligned}
 3. \quad (a) \quad y &= \frac{(2x-3)^3}{(x+5)^4} \\
 &= \frac{[2(u-5)-3]^3}{u^4} \\
 &= \frac{(2u-13)^3}{u^4}
 \end{aligned}$$

1A

$$\begin{aligned}
 (b) \quad y &= \frac{(2u-13)^3}{u^4} \\
 &= \frac{8u^3 - 156u^2 + 1014u - 2197}{u^4} \\
 &= 8u^{-1} - 156u^{-2} + 1014u^{-3} - 2197u^{-4} \\
 \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}
 \end{aligned}$$

1M

$$\begin{aligned}
 &= (-8u^{-2} + 312u^{-3} - 3042u^{-4} + 8788u^{-5}) (1) \\
 &= -8u^{-2} + 312u^{-3} - 3042u^{-4} + 8788u^{-5}
 \end{aligned}$$

1A

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= \frac{d}{du} (-8u^{-2} + 312u^{-3} - 3042u^{-4} + 8788u^{-5}) \cdot \frac{du}{dx} \\
 &= 16u^{-3} - 936u^{-4} + 12168u^{-5} - 43940u^{-6} \\
 &= \frac{16}{(x+5)^3} - \frac{936}{(x+5)^4} + \frac{12168}{(x+5)^5} - \frac{43940}{(x+5)^6}
 \end{aligned}$$

1M
1

$$\begin{aligned}
 4. \quad (a) \quad y &= \frac{(6x+5)^3}{(3x+1)^2} \\
 &= \frac{[2(u-1)+5]^3}{u^2} \\
 &= \frac{(2u+3)^3}{u^2}
 \end{aligned}$$

1A

$$\begin{aligned}
 (b) \quad y &= \frac{8u^3 + 36u^2 + 54u + 27}{u^2} \\
 &= 8u + 36 + 54u^{-1} + 27u^{-2} \\
 \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}
 \end{aligned}$$

1M

$$\begin{aligned}
 &= (8 - 54u^{-2} - 54u^{-3}) (3) \\
 &= 24 - 162u^{-2} - 162u^{-3}
 \end{aligned}$$

1A

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= \frac{d}{dx} (24 - 162u^{-2} - 162u^{-3}) \frac{du}{dx} \\
 &= (324u^{-3} + 486u^{-4}) (3) \\
 &= 972u^{-3} + 1458u^{-4} \\
 &= \frac{972u + 1458}{u^4} \\
 &= \frac{972(3x+1) + 1458}{(3x+1)^4} \\
 &= \frac{486(6x+5)}{(3x+1)^4}
 \end{aligned}$$

1A

5. (a) $x^2 + y^2 = 30$
- $$2x + 2y \frac{dy}{dx} = 0 \quad 1M$$
- $$\frac{dy}{dx} = -\frac{x}{y} \quad 1A$$
- $$\left. \frac{dy}{dx} \right|_{(p, q)} = \frac{-p}{q} = -\frac{1}{2}$$
- $$q = 2p \quad 1$$
- (b) $p^2 + (2p)^2 = 30$ 1M
- $$p^2 = 6$$
- $$p = \pm\sqrt{6}$$
- Thus, $(p, q) = (\sqrt{6}, 2\sqrt{6})$ or $(-\sqrt{6}, -2\sqrt{6})$. 1A+1A

6. $\frac{d}{dx} [\sin x + \sin 3x + \dots + \sin(2n-1)x] = \frac{d}{dx} \left[\frac{1 - \cos 2nx}{2 \sin x} \right]$ 1M
- $$\cos x + 3 \cos 3x + \dots + (2n-1) \cos(2n-1)x = \frac{(2n \sin 2nx)(\sin x) - (1 - \cos 2nx)(\cos x)}{2 \sin^2 x} \quad 1M$$
- Put $n = 18$ and $x = \frac{\pi}{6}$,
- $$\begin{aligned} & \cos \frac{\pi}{6} + 3 \cos \frac{3\pi}{6} + \dots + 35 \cos \frac{35\pi}{6} \\ &= \frac{36 \sin(6\pi) \left(\sin \frac{\pi}{6} \right) - (1 - \cos 6\pi) \left(\cos \frac{\pi}{6} \right)}{2 \sin^2 \frac{\pi}{6}} \quad 1M \\ &= \frac{0 - 0}{2 \left(\frac{1}{2} \right)^2} \\ &= 0 \quad 1A \end{aligned}$$

$$\begin{aligned}
7. \quad (a) \quad y &= \frac{\sin^2 x + \csc^2 x}{\sin^2 x - \csc^2 x} \\
&= \frac{u + \frac{1}{u}}{u - \frac{1}{u}} \times \frac{u}{u} & 1M \\
&= \frac{u^2 + 1}{u^2 - 1} \\
&= 1 + \frac{2}{u^2 - 1} \\
\frac{dy}{du} &= 2(u^2 - 1)^{-2}(2u) & 1M \\
&= \frac{4u}{(u^2 - 1)^2} & 1A \\
(b) \quad \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} & 1M \\
&= -\frac{4u}{(u^2 - 1)^2} \cdot (2 \sin x \cos x) \\
&= -\frac{4 \sin^2 x (2 \sin x \cos x)}{(\sin^2 x + 1)^2 (\sin^2 x - 1)^2} \\
&= -\frac{8 \sin^3 x \cos x}{(\sin^2 x + 1)^2 (-\cos^2 x)^2} \\
&= -\frac{8 \tan^3 x}{(\sin^2 x + 1)^2} & 1
\end{aligned}$$

$$\begin{aligned}
8. \quad \sin xy &= \sec(x + y)^2 \\
\sin xy &= [\cos(x + y)^2]^{-1} & 1M \\
\cos xy \left(y + x \frac{dy}{dx} \right) &= -[\cos(x + y)^2]^{-2} [-\sin(x + y)^2] (2)(x + y) \left(1 + \frac{dy}{dx} \right) & 1M \\
[x \cos xy - 2(x + y) \sin(x + y)^2 \sec^2(x + y)^2] \frac{dy}{dx} &= 2(x + y) \sin(x + y)^2 \sec^2(x + y)^2 - y \cos xy \\
\frac{dy}{dx} &= \frac{2(x + y) \sin(x + y)^2 \sec^2(x + y)^2 - y \cos xy}{x \cos xy - 2(x + y) \sin(x + y)^2 \sec^2(x + y)^2} & 1A
\end{aligned}$$

$$\begin{aligned}
9. \quad \sin(x^2 + y) &= \tan(x + 1) - 1 \\
\cos(x^2 + y) \left(2x + \frac{dy}{dx} \right) &= \sec^2(x + 1) & 1M \\
\cos(x^2 + y) \frac{dy}{dx} &= \sec^2(x + 1) - 2x \cos(x^2 + y) \\
\frac{dy}{dx} &= \frac{\sec^2(x + 1) - 2x \cos(x^2 + y)}{\cos(x^2 + y)} & 1A
\end{aligned}$$

10. Note that in the domain of the function, we have $\cot x^2 > 0$, i.e., $\tan x^2 > 0$.

$$\begin{aligned}
 y &= \left[\ln(\tan^2 x^2) \right] \left(\ln \sqrt{\cot x^2} \right) \\
 &= 2 \ln(\tan x^2) \cdot \left(-\frac{1}{2} \right) \ln(\tan x^2) \\
 &= - \left[\ln(\tan x^2) \right]^2 & 1A \\
 \frac{dy}{dx} &= -2 \left[\ln(\tan x^2) \right] \left(\frac{1}{\tan x^2} \right) (\sec^2 x^2)(2x) & 1M \\
 &= -4x \csc x^2 \sec x^2 \ln(\tan x^2) & 1A
 \end{aligned}$$

11. $\ln(x^2 y) = 4x^3 - 10y$

$$\begin{aligned}
 \frac{1}{x^2 y} \left(2xy + x^2 \frac{dy}{dx} \right) &= 12x^2 - 10 \frac{dy}{dx} & 1M \\
 \left(\frac{1}{y} + 10 \right) \frac{dy}{dx} &= 12x^2 - \frac{2}{x} \\
 \left(\frac{1 + 10y}{y} \right) \frac{dy}{dx} &= \frac{12x^3 - 2}{x} \\
 \frac{dy}{dx} &= \frac{2y(6x^3 - 1)}{x(1 + 10y)} & 1A
 \end{aligned}$$

12. $e^{3x-2y} + x = y - \cos y$

$$\begin{aligned}
 e^{3x-2y} \left(3 - 2 \frac{dy}{dx} \right) + 1 &= \frac{dy}{dx} + \sin y \frac{dy}{dx} & 1M \\
 \left(-2e^{3x-2y} - 1 - \sin y \right) \frac{dy}{dx} &= -3e^{3x-2y} - 1 \\
 \frac{dy}{dx} &= \frac{3e^{3x-2y} + 1}{2e^{3x-2y} + 1 + \sin y} & 1A
 \end{aligned}$$

13. $y' = x^k \left(\frac{1}{x} \right) + kx^{k-1} \ln x$

$$\begin{aligned}
 &= x^{k-1} (1 + k \ln x) & 1A \\
 y'' &= (k-1)x^{k-2} (1 + k \ln x) + x^{k-1} \left(\frac{k}{x} \right) \\
 &= x^{k-2} [2k - 1 + k(k-1) \ln x] & 1A \\
 4x^2 y'' - 8xy' + 9y &= 0 \\
 4x^k [2k - 1 + k(k-1) \ln x] - 8x^k (1 + k \ln x) + 9x^k \ln x &= 0 \\
 x^k [(4k^2 - 12k + 9) \ln x + 8k - 12] &= 0 \\
 x^k [(2k-3)^2 \ln x + 4(2k-3)] &= 0 \\
 x^k (2k-3) [(2k-3) \ln x + 4] &= 0 & 1M \\
 2k - 3 &= 0 \\
 k &= \frac{3}{2} & 1A
 \end{aligned}$$

14. (a) $y^2 = x + x^{\frac{1}{2}}$
 $2yy' = 1 + \frac{1}{2}x^{-\frac{1}{2}}$ 1M
 $4yy' - x^{-\frac{1}{2}} = 2$ 1
- (b) $4yy' - x^{-\frac{1}{2}} = 2$
 $4(y')^2 + 4yy'' + \frac{1}{2}x^{-\frac{3}{2}} = 0$ 1M
 $yy'' + (y')^2 + \frac{1}{8}x^{-\frac{3}{2}} = 0$ 1
15. (a) $y = x^2e^x$
 $y' = 2xe^x + x^2e^x$ 1A
 $y'' = (2e^x + 2xe^x) + (2xe^x + x^2e^x)$
 $= 2e^x + 4xe^x + x^2e^x$ 1A
- (b) $y'' + y' - 2y = (2e^x + 4xe^x + x^2e^x) + (2xe^x + x^2e^x) - 2x^2e^x$ 1M
 $= e^x[2 + (4x + 2x) + (x^2 + x^2 - 2x^2)]$
 $= e^x(6x + 2)$
Thus, $a = 6$ and $b = 2$. 1A+1A
16. $\frac{dy}{dx} = 2n(2x + 3)^{n-1}$ 1A
 $\frac{d^2y}{dx^2} = 4n(n-1)(2x + 3)^{n-2}$ 1A
Since $(2x + 3)^2 \frac{d^2y}{dx^2} - 9(2x + 3) \frac{dy}{dx} + 10y = 0$ for all real values of x ,
 $4n(n-1)(2x + 3)^n - 18n(2x + 3)^n + 10(2x + 3)^n = 0$ 1M
 $(2x + 3)^n(4n^2 - 22n + 10) = 0$
 $n = 5 \quad \text{or} \quad \frac{1}{2} \text{ (rejected)}$ 1A