Dexter Wong & His Mathematics Team

Summer Course 2022 – 2023

MATHEMATICS Compulsory Part

S5 – S6 Core Assignment Set 2

Name: ____________________ Telephone: ____________

Student number: ____________________ Centre: ____________

Lesson date: SUN/MON/TUE/WED/THU/FRI/SAT

INSTRUCTIONS

1. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
2. Unless otherwise specified, all workings (except for multiple choice questions) must be clearly shown.
3. Unless otherwise specified, numerical answers should be either exact or correct to 3 significant figures.
4. The diagrams in this paper are not necessarily drawn in scale.

Suggested solution

Distributed in summer course
S5 – S6 Core
Phase 1 – Lesson 3

1. B

Since OA is horizontal, the altitude through B is $x = 20$.

The coordinates of the orthocentre H are $(20,\ 4k - 5)$.

Since $OH \perp AB$,

$$\frac{4k-5}{20} \times \frac{2k-0}{20-25} = -1$$

$$8k^2 - 10k - 100 = 0$$

Required sum = sum of roots = $\frac{10}{8} = \frac{5}{4}$

2. D

The straight line $5x + 2y = 2b$ passes through $B\,(0,\ b)$.

Slope of straight line $5x + 2y = 2b$ is $-\frac{5}{2}$, and the inclination is around $112°$.

Denote the angle between the line $5x + 2y = 2b$ and the y-axis by θ. We have $\theta \approx 21.8°$.

$\angle OBA = 2\theta \approx 43.6°$ and $\tan(2\theta) = \frac{a}{b} = \frac{20}{21}$.

So, $a : b = 20 : 21$.

3. B

The coordinates of the vertices of the triangle are $(0,\ 0)$, $(6,\ 0)$ and $(0,\ 8)$.

Let the radius of inscribed circle be r.

By considering the area of the triangle,

$$\frac{(6)(8)}{2} = \frac{(6)(r)}{2} + \frac{(8)(r)}{2} + \frac{(\sqrt{6^2+8^2})(r)}{2}$$

$$r = 2$$

The coordinates of the incentre are $(2,\ 2)$.

4. B

Circumcentre lies on the perpendicular bisector of $AC \Rightarrow$ it lies on $x = \frac{2+(-4)}{2} = -1$

Let the coordinates of circumcentre be $(-1,\ k)$.

$$\sqrt{(-1-7)^2 + (k-5)^2} = \sqrt{(-1-2)^2 + (k+6)^2}$$

$$k^2 - 10k + 89 = k^2 + 12k + 45$$

$$k = 2$$

5. D

$OE = \sqrt{8^2 + 15^2} = 17$ and $EF = \sqrt{20^2 + 15^2} = 25$

Let the radius of inscribed circle be r. Consider the area of $\triangle OEF$,

$$\frac{(28-0)(15)}{2} = \frac{17r}{2} + \frac{25r}{2} + \frac{28r}{2}$$
$$r = 6$$

y-coordinate of G is 6. Let the x-coordinate of G be g.

$\tan \angle FOE = \text{slope of } OE \quad \Rightarrow \quad \angle FOE = \tan^{-1}\frac{15}{8}$

$$\text{Slope of } OG = \frac{6}{g} = \tan\frac{\angle FOE}{2}$$
$$= \frac{3}{5} \qquad \textit{(by calculator)}$$
$$g = 10$$

I. ✓.

II. ✓.

III. ✓. $m = \dfrac{6-0}{10-28} = -\dfrac{1}{3}$ and $n = \dfrac{15-0}{8-28} = -\dfrac{3}{4}$

$$\frac{2m}{1-m^2} = -\frac{3}{4} = n$$

6. C

The straight line $x - 2y + 10 = 0$ is perpendicular to the straight line $2x + y + a = 0$.

The triangle formed is a right-angled triangle. The orthocentre lies at the right-angled vertex.

When $x = -6$,

$$(-6) - 2y + 10 = 0$$
$$y = 2$$

Substitute $(-6,\ 2)$ into $2x + y + a = 0$,

$$2(-6) + (2) + a = 0$$
$$a = 10$$

7. D

$x\text{-coordinate of vertex} = \dfrac{-k}{2(1)} = -\dfrac{k}{2}$

$\text{Midpoint of } PR = \left(-\dfrac{k}{2},\ 0\right)$

Consider the x-coordinate of centroid,

$$-2 = \frac{0(1) + \left(-\frac{k}{2}\right)(2)}{1+2}$$
$$k = 6$$

8. $\boxed{\text{B}}$

$\triangle OAB$ is a right-angled triangle. So, orthocentre of $\triangle OAB$ is at point O.

Area of $\triangle OAB = \frac{1}{2}(12)(16) = 96$.

$AB = \sqrt{12^2 + 16^2} = 20$. Let the required distance be d. By considering the area of $\triangle OAB$,

$$\frac{1}{2}(AB)(d) = 96$$
$$d = 9.6$$

9. $\boxed{\text{B}}$

$$\sqrt{(-4+9)^2 + (2+2)^2} = \sqrt{(-4-0)^2 + (k-2)^2}$$
$$(k-2)^2 = 25$$
$$k = -3 \quad \text{or} \quad 7$$

10. $\boxed{\text{B}}$

$OB = \sqrt{3^2 + 4^2} = 5 = OA$

Since $\triangle OAB$ is an isosceles triangle, the perpendicular bisector of AB passes through circumcentre and O.

Slope of $AB = \frac{4-0}{3+5} = \frac{1}{2}$ and slope of perpendicular bisector $= -2$.

$$-2 = \frac{k-0}{h-0}$$
$$h = -\frac{k}{2}$$

11. $\boxed{\text{B}}$

Altitude through P is the x-axis.

The coordinates of the orthocentre H are $(6,\ 0)$.

Let the coordinates of R be $(0,\ r)$. Since $RH \perp PQ$,

$$\frac{0-r}{6-0} \times \frac{-3}{4} = -1$$
$$r = -8$$

The coordinates of R are $(0,\ -8)$.

12. $\boxed{\text{D}}$

y-coordinate of circumcentre $= \frac{(-2)+(8)}{2} = 3$

Let the coordinates of the circumcentre be $(h,\ 3)$.

$$\sqrt{(h-2)^2 + (3-8)^2} = \sqrt{(h-10)^2 + (14-3)^2}$$
$$h^2 - 4h + 29 = h^2 - 20h + 221$$
$$16h = 192$$
$$h = 12$$

x-coordinate of the circumcentre is 12.

13. A

Since AB is parallel to the y-axis, altitude through O is parallel to the x-axis.
Let the coordinates of orthocentre H be $(x,\ 0)$. Since $AH \perp OB$,

$$\frac{12-0}{16-x} \times \frac{-12-0}{16-0} = -1$$
$$x = 7$$

14. D

Note that the altitude passing through B is horizontal.
The y-coordinate of the orthocentre is 0.
Let the coordinates of the orthocentre H be $(h,\ 0)$.
Since $CH \perp AB$,

$$\frac{6-0}{0-h} \times \frac{0+2}{2-0} = -1$$
$$h = 6$$

x-coordinate of the orthocentre is 6.

15. A

Let the radius of the inscribed circle be r.

$$\frac{(9)(12)}{2} = \frac{9r}{2} + \frac{12r}{2} + \frac{\sqrt{9^2+12^2}\,r}{2}$$
$$r = 3$$

Required coordinates are $(3,\ 3)$.

16. C

Denote the in-centre of $\triangle OAB$ by I.

Slope of $AI = \dfrac{1-0}{1-3} = -\dfrac{1}{2}$

$-\tan\angle OAI = -\dfrac{1}{2}$

$\angle OAI = \tan^{-1}\dfrac{1}{2}$

$\tan\angle OAB = \tan(2\angle OAI) = \dfrac{4}{3}$ (by calculator)

$\dfrac{OB}{OA} = \tan\angle OAB$

$\dfrac{OB}{3} = \dfrac{4}{3}$

$OB = 4$

Required y-coordinate is 4.

17. $\boxed{\text{C}}$

Let the centre of circle be O.

BO is a median of $\triangle ABC$. So, BEO is a straight line.

$\angle BAC = \angle ABE = 27^\circ$

$\angle CBD = \angle BAC = 27^\circ$

$\angle ABC = 90^\circ$

In $\triangle ABD$,

$$\begin{aligned} x + 27^\circ + (90^\circ + 27^\circ) &= 180^\circ \\ x &= 36^\circ \end{aligned}$$

18. $\boxed{\text{D}}$

Let $H\,(h,\ k)$ be the orthocentre.

$AH \perp BC$ and $BH \perp AC$

$$\frac{k+19}{h+38} \times \frac{9+1}{-10+2} = -1 \qquad \frac{k-9}{h+10} \times \frac{-1+19}{-2+38} = -1$$

$$\frac{k+19}{h+38} = \frac{4}{5} \qquad \frac{k-9}{h+10} = -2$$

Solving, we have $h = -8$ and $k = 5$.

19. $\boxed{\text{C}}$

RT is the angle bisector of $\angle SRP$. So, $\angle SRP = 24^\circ$.

$\angle QPS = \angle QSR$.

$$\begin{aligned} \angle QPS + (\angle QSR + 70^\circ) + 24^\circ &= 180^\circ \\ \angle QPS &= 43^\circ \end{aligned}$$

20. $\boxed{\text{A}}$

$P\left(\frac{k}{2},\ 0\right)$, $Q\,(0,\ -k)$ and $R\,(0,\ k)$.

Circumcentre lies on perpendicular bisector of QR, i.e., the x-axis.

Circumcentre is at $(-3,\ 0)$.

$$\begin{aligned} \frac{k}{2} - (-3) &= \sqrt{3^2 + k^2} \\ 0 &= \frac{3k^2}{4} - 3k \\ k &= 4 \quad \text{or} \quad 0 \end{aligned}$$

y-coordinate of $R = 4$

21. (a) k 1A

(b) (i) Note that $OH \perp PQ$.

$$\frac{k-0}{h-0} \times \frac{k-60}{96-0} = -1$$ 1M

$$k^2 - 60k + 96h = 0$$ 1

(ii) $h = -\frac{1}{96}k^2 + \frac{5}{8}k$

$$= -\frac{1}{96}\left[k^2 - 2(30)k + (30)^2\right] + \frac{75}{8}$$ 1M

$$= -\frac{1}{96}(k-30)^2 + \frac{75}{8}$$

Required value is $\frac{75}{8}$. 1A

Alternative solution

$\Delta = 60^2 - 4(1)(96h) \geq 0$ (1M)

$$h \leq \frac{75}{8}$$

Required value is $\frac{75}{8}$. (1A)

22. (a) $a = \frac{(PQ)r}{2} + \frac{(QR)r}{2} + \frac{(PR)r}{2}$ 1M

$$= \frac{r}{2}(PQ + QR + PR)$$

$$= \frac{pr}{2}$$ 1

(b) $AB = \sqrt{(9-2)^2 + (18+6)^2} = 25$, $BC = \sqrt{32^2 + 24^2} = 40$ and $AC = 39$.

Let the radius of the inscribed circle of $\triangle ABC$ be r.

$$\frac{(41-2)(18+6)}{2} = \frac{(25+40+39)r}{2}$$ 1M

$$r = 9$$ 1A

Required y-coordinate $= -6 + 9 = 3$ 1A

23. (a) $\angle OMQ = 90^\circ = \angle ONQ$ *(given)*

$AB = CD$ *(given)*

$OM = ON$ *(equal chords, equidistant from centre)*

$OQ = OQ$ *(common side)*

$\triangle QNO \cong \triangle QMO$ *(RHS)*

Marking Scheme		
Case 1	Any correct proof with correct reasons.	2
Case 2	Any correct proof without reasons.	1

(b) (i) $OT = ON = 130$

$OQ = \sqrt{312^2 + 130^2} = 338$ and $OT : OQ = 130 : 338 = 5 : 13$

x-coordinate of $T = 312 \times \left(-\frac{5}{13}\right) = -120$ 1M

y-coordinate of $T = -130 \times \left(-\frac{5}{13}\right) = 50$

The coordinates of T are $(-120, 50)$. 1A

Let the coordinates of R be $(h, -130)$ such that $QR \perp ON$.

$$\frac{-130-50}{h+120} \times \frac{50-0}{-120-0} = -1$$ 1M

$$h = -195$$

Let the coordinates of P be (a, b). Note that P is the midpoint of PR.

$$\frac{a+(-195)}{2} = -120 \quad \text{and} \quad \frac{b+(-130)}{2} = 50$$ 1M

$$a = -45 \qquad b = 230$$

The coordinates of P are $(-45, 230)$. 1A

(ii) $OP = \sqrt{45^2 + 230^2} = \sqrt{54\,925} \neq OQ$ 1M

Thus, O is not the circumcentre of $\triangle PQR$.

The claim is disagreed. 1A

24. (a) $PB = PD$ and $\angle PBD = \dfrac{180° - x}{2} = 90° - \dfrac{x}{2}$ 1M

$\angle BAD = \angle PBD = 90° - \dfrac{x}{2}$ 1A

$\angle ABC = 90°$

$\angle AQB = 180° - 90° - \left(90° - \dfrac{x}{2}\right) = \dfrac{x}{2}$ 1A

(b) (i) Since $PB = PD = PR$, we have $\angle BRD = \angle PDR$. 1M

$$\angle BRD + \angle PDR = x$$

$$\angle BRD = \frac{x}{2}$$ 1A

(ii) Note that $\angle BRD = \angle BQD = \dfrac{x}{2}$.

B, D, Q, R are concyclic. 1M

Since $PB = PD = PR$, P is the centre of the circle BDR,

and hence is the centre of the circle $BDQR$.

Thus, P is the centre of the circumcircle of $\triangle BDQ$.

The claim is agreed. 1A

25. (a) $CE \perp AB$ *(property of orthocentre)*

$BD \perp AC$ *(property of orthocentre)*

$\angle BEC = \angle BDC = 90°$

Thus, $BCDE$ is a cyclic quadrilateral. *(converse of $\angle$s in the same segment)*

Marking Scheme		
Case 1	Any correct proof with correct reasons.	2
Case 2	Any correct proof without reasons.	1

(b) (i) Coordinates of centre $= \left(\dfrac{-6+14}{2}, \dfrac{-6-6}{2}\right) = (4, -6)$ 1A

The equation of the circle is

$$(x-4)^2 + (y+6)^2 = (0-4)^2 + (8+6)^2$$ 1M

$$(x-4)^2 + (y+6)^2 = 100$$ 1A

(ii) Distance between A and centre $= \sqrt{4^2 + (-6-8)^2} = \sqrt{212}$

Radius of circle = 10

Angle between two tangents $= 2 \times \tan^{-1} \dfrac{10}{\sqrt{212}} \approx 86.8° \neq 90°$ 1M+1A

The claim is not agreed. 1A

26. (a) AD is the perpendicular bisector of BC. The coordinates of D are $(-8,\ -6)$. 1A

Since D is the midpoint of BC, the coordinates of C are $(-8,\ -16)$. 1A

Midpoint of $AC = (1,\ -11)$ and slope of $AC = \dfrac{-6+16}{10+8} = \dfrac{5}{9}$ 1A

Required equation is

$$y + 11 = -\frac{9}{5}(x-1) \quad \text{1M}$$

$$9x + 5y + 46 = 0 \quad \text{1A}$$

(b) Since AD is perpendicular bisector of BC, y-coordinate of circumcentre $= -6$. 1M

Put $y = -6$ into $9x + 5y + 46 = 0$, we have $x = -\dfrac{16}{9}$.

Required coordinates are $\left(-\dfrac{16}{9},\ -6\right)$. 1A

(c) Required equation is

$$\left(x + \frac{16}{9}\right)^2 + (y+6)^2 = \left(10 + \frac{16}{9}\right)^2 + (-6+6)^2 \quad \text{1M}$$

$$\left(x + \frac{16}{9}\right)^2 + (y+6)^2 = \frac{11\,236}{81} \quad \text{1A}$$

(d) (i) Locus of P is a parabola (opens rightwards). 1A

(ii) Let the coordinates of P be $(x,\ y)$.

$$\sqrt{(x-10)^2 + (y+6)^2} = \sqrt{(x+8)^2} \quad \text{1M}$$

$$y^2 - 36x + 12y + 72 = 0$$

Required equation is $y^2 - 36x + 12y + 72 = 0$. 1A

27. (a) Let $x = \angle BAI$ and $y = \angle ABI$.

$\angle PAC = \angle BAI = x$ *(property of incentre)*

$PB = PC$ *(equal $\angle$s, equal chords)*

$\angle PIB = \angle ABI + \angle BAI$ *(ext. $\angle$ of $\triangle$)*

$= x + y$

$\angle IBC = \angle ABI = y$ *(property of incentre)*

$\angle PBC = \angle PAC = x$ *($\angle$s in the same segment)*

$\angle IBP = x + y$

$= \angle PIB$

$PB = PI$ *(sides opp. equal $\angle$s)*

Thus, $PB = PI = PC$.

Marking Scheme		
Case 1	Any correct proof with correct reasons.	3
Case 2	Any correct proof without reasons.	2
Case 3	Incomplete proof with any one correct step with reason.	1

(b) $\angle IAY = \angle PSC$ *($\angle$s in the same segment)*

$\angle AYI = 90^\circ$ *(given)*

$\angle SCP = 90^\circ$ *($\angle$ in semicircle)*

$= \angle AYI$

$\triangle IAY \sim \triangle PSC$ *(AA)*

Marking Scheme		
Case 1	Any correct proof with correct reasons.	3
Case 2	Any correct proof without reasons.	2
Case 3	Incomplete proof with any one correct step with reason.	1

(c) $\dfrac{IY}{PC} = \dfrac{IA}{PS}$ 1M

$\dfrac{r}{IP} = \dfrac{AI}{2R}$

$AI \cdot IP = 2Rr$

The claim is agreed. 1A

(d) Let the equation of circle BPC be $x^2 + y^2 + Dx + Ey + F = 0$, where D, E and F are constants.

$$\begin{cases} (-16)^2 - 16D + F = 0 \\ (-8)^2 - 8E + F = 0 \\ 16^2 + 16D + F = 0 \end{cases}$$ 1M

Solving, we have $D = 0$, $E = -24$ and $F = -256$.

Radius of the circumcircle $= \sqrt{0^2 + 12^2 + 256} = 20$ 1A

$BP = \sqrt{16^2 + 8^2} = 8\sqrt{5}$ 1A

By (c), $AI = \dfrac{2Rr}{IP} = \dfrac{2(20)(9)}{8\sqrt{5}} = 9\sqrt{5}$ 1A

END OF PAPER