

Dexter Wong & His Mathematics Team

暑期課程 2022 – 2023

數學 延伸部分

S5 – S6 M2 積分 功課 第 2 套

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上課日：日 / 一 / 二 / 三 / 四 / 五 / 六

考生須知

1. 本試卷各題均須作答，答案須寫在本試題答題簿中預留的空位內。
2. 除特別指明外，須詳細列出所有算式。
3. 除特別指明外，數值答案須用真確值表示。
4. 本試卷的附圖不一定依比例繪成。

建議題解



派發於暑期課程
S5 – S6 M2 積分
第 1 期 – 第 2 堂

1. (a) 設 $u = x + 2$ 。則 $du = dx$ 。 1M

$$\begin{aligned}\int \frac{x-1}{x+2} dx &= \int \frac{u-3}{u} du \\ &= \int \left(1 - \frac{3}{u}\right) du \\ &= u - 3 \ln |u| + \text{常數} \\ &= x - 3 \ln |x+2| + \text{常數}\end{aligned}$$

1A

- (b) 設 $u = \sqrt{x}$ 。則 $du = \frac{1}{2\sqrt{x}} dx$ 。 1M

$$\begin{aligned}\int \frac{\sqrt{x}-1}{x+2\sqrt{x}} dx &= 2 \int \left(\frac{\sqrt{x}-1}{\sqrt{x}+2} \cdot \frac{1}{2\sqrt{x}}\right) dx \\ &= 2 \int \frac{u-1}{u+2} du \\ &= 2u - 6 \ln |u+2| + \text{常數} \\ &= 2\sqrt{x} - 6 \ln(\sqrt{x}+2) + \text{常數}\end{aligned}$$

1A
1A
1A

2. (a) 設 $u = x - 1$ 。則 $du = dx$ 。 1M

$$\begin{aligned}\int x(x-1)^n dx &= \int (u+1)u^n du \\ &= \int (u^{n+1} + u^n) du \\ &= \frac{(x-1)^{n+2}}{n+2} + \frac{(x-1)^{n+1}}{n+1} + \text{常數}\end{aligned}$$

1A
1A

- (b) 設 $x = \sin \theta$ 。則 $dx = \cos \theta d\theta$ 。 1M

$$\begin{aligned}\int \sin 2\theta (\sin \theta - 1)^{2014} d\theta &= 2 \int \sin \theta \cos \theta (\sin \theta - 1)^{2014} d\theta \\ &= 2 \int x(x-1)^{2014} dx \\ &= \frac{(\sin \theta - 1)^{2016}}{1008} + \frac{2(\sin \theta - 1)^{2015}}{2015} + \text{常數}\end{aligned}$$

1A

3. (a) $\frac{\cot^2 \theta - 1}{1 + \cot^2 \theta} = \frac{\cos^2 \theta - \sin^2 \theta}{\sin^2 \theta + \cos^2 \theta}$ 1M
 $= \frac{\cos 2\theta}{1} = \cos 2\theta$ 1

(b) $\int \frac{\sin \theta \cos \theta (1 + \cot^2 \theta)}{\cot^2 \theta - 1} d\theta = \int \frac{\sin \theta \cos \theta}{\cos 2\theta} d\theta$ 1M

$$\begin{aligned}&= \frac{1}{2} \int \frac{\sin 2\theta}{\cos 2\theta} d\theta \\ &= -\frac{1}{4} \int \frac{du}{u} \quad (\text{其中 } u = \cos 2\theta) \\ &= -\frac{1}{4} \ln |\cos 2\theta| + \text{常數}\end{aligned}$$

1M
1A

4. (a) 設 $u = x^2 - 8x + 32$ 。則 $du = (2x - 8) dx$ 。 1M

$$\begin{aligned}\int \frac{x-4}{x^2-8x+32} dx &= \frac{1}{2} \int \frac{du}{u} & 1M \\ &= \frac{\ln|u|}{2} + \text{常數} \\ &= \frac{\ln|x^2-8x+32|}{2} + \text{常數} & 1A\end{aligned}$$

$$\begin{aligned}\text{(b)} \int \frac{x^2 dx}{x^2-8x+32} &= \int \left(1 + \frac{8x-32}{x^2-8x+32}\right) dx & 1M \\ &= x + 4 \ln|x^2-8x+32| + \text{常數} & 1A\end{aligned}$$

5. (a) 設 $u = x^2 + 2x + 2$ 。則 $du = (2x + 2) dx$ 。 1M

$$\begin{aligned}\int \frac{2x+2}{x^2+2x+2} dx &= \int \frac{du}{u} & 1M \\ &= \ln|u| + \text{常數} \\ &= \ln|x^2+2x+2| + \text{常數} & 1A\end{aligned}$$

$$\begin{aligned}\text{(b)} \int \frac{x^2}{x^2+2x+2} dx &= \int \left(1 - \frac{2x+2}{x^2+2x+2}\right) dx & 1M \\ &= x - \ln|x^2+2x+2| + \text{常數} & 1A\end{aligned}$$

6. 設 $u = 1 + \frac{3}{x^2}$ 。則 $du = -\frac{6}{x^3} dx$ 。

$$\begin{aligned}\int \frac{\sqrt{x^2+3}}{x^4} dx &= -\frac{1}{6} \int \left(\frac{\sqrt{x^2+3}}{x}\right) \left(-\frac{6}{x^3}\right) dx \\ &= -\frac{1}{6} \int \sqrt{1+\frac{3}{x^2}} \left(-\frac{6}{x^3}\right) dx & 1M \\ &= -\frac{1}{6} \int \sqrt{u} du & 1A \\ &= -\frac{1}{9} u^{\frac{3}{2}} + \text{常數} \\ &= -\frac{1}{9} \left(1 + \frac{3}{x^2}\right)^{\frac{3}{2}} + \text{常數} & 1A\end{aligned}$$

7. 設 $u = 3 + \sin(\ln x)$ 。則 $du = \frac{\cos(\ln x)}{x} dx$ 。 1M

$$\begin{aligned}\int \frac{\cos(\ln x)}{x[3 + \sin(\ln x)]} dx &= \int \frac{du}{u} & 1A \\ &= \ln|3 + \sin(\ln x)| + \text{常數} \\ &= \ln(3 + \sin(\ln x)) + \text{常數} & 1A\end{aligned}$$

8. 設 $u = \sin^2 x$ 。則 $du = 2 \sin x \cos x \, dx$ 。 1A

$$\begin{aligned} \int \frac{\sin x \cos x}{\sqrt{9 \sin^2 x + 4 \cos^2 x}} \, dx &= \frac{1}{2} \int \frac{du}{\sqrt{9u + 4(1-u)}} & 1A \\ &= \frac{1}{2} \int \frac{du}{\sqrt{5u + 4}} \\ &= \frac{\sqrt{5u + 4}}{5} + \text{常數} \\ &= \frac{\sqrt{5 \sin^2 x + 4}}{5} + \text{常數} & 1A \end{aligned}$$

$$\begin{aligned} 9. \quad (a) \quad \int \sqrt{1 + \cos x} \, dx &= \int \sqrt{1 + \left(2 \cos^2 \frac{x}{2} - 1\right)} \, dx & 1M \\ &= \sqrt{2} \int \cos \frac{x}{2} \, dx \\ &= 2\sqrt{2} \sin \frac{x}{2} + \text{常數} & 1A \end{aligned}$$

(b) 設 $u = \frac{\pi}{2} - x$ 。則 $du = -dx$ 。

$$\begin{aligned} \int \sqrt{1 + \sin x} \, dx &= - \int \sqrt{1 + \sin \left(\frac{\pi}{2} - u\right)} \, du & 1A \\ &= - \int \sqrt{1 + \cos u} \, du \\ &= -2\sqrt{2} \sin \frac{u}{2} + \text{常數} & 1M \\ &= -2\sqrt{2} \sin \left(\frac{\pi}{4} - \frac{x}{2}\right) + \text{常數} & 1A \end{aligned}$$

10. (a) 設 $u = x^3$ 。則 $du = 3x^2 \, dx$ 。 1M

$$\begin{aligned} \int x^2 2^{x^3} \, dx &= \frac{1}{3} \int 2^u \, du \\ &= \frac{1}{3} \int e^{u \ln 2} \, du \\ &= \frac{e^{u \ln 2}}{3 \ln 2} + \text{常數} & 1A \\ &= \frac{2^u}{3 \ln 2} + \text{常數} \\ &= \frac{2^{x^3}}{3 \ln 2} + \text{常數} & 1A \end{aligned}$$

(b) 設 $u = x^3$ 。則 $du = 3x^2 \, dx$ 。

$$\begin{aligned} \int x^2 (2^{x^3} + 3^{x^3} + 5^{x^3}) \, dx &= \frac{1}{3} (2^u + 3^u + 5^u) \, du & 1M \\ &= \frac{2^u}{3 \ln 2} + \frac{3^u}{3 \ln 3} + \frac{5^u}{3 \ln 5} + \text{常數} & 1M \\ &= \frac{2^{x^3}}{3 \ln 2} + \frac{3^{x^3}}{3 \ln 3} + \frac{5^{x^3}}{3 \ln 5} + \text{常數} & 1A \end{aligned}$$

11. (a) 設 $u = 1 + 4 \sin^2 x$ 。則 $du = 8 \sin x \cos x \, dx$ 。 1M

$$\int \frac{\sin 2x}{1 + 4 \sin^2 x} \, dx = \frac{1}{4} \int \frac{8 \sin x \cos x}{1 + 4 \sin^2 x} \, dx$$

$$= \frac{1}{4} \int \frac{du}{u}$$

1A

$$= \frac{1}{4} \ln(1 + 4 \sin^2 x) + \text{常數}$$

1A

- (b) 設 $u = \sin^2 x - 3 \sin x + 2$ 。則 $du = (2 \sin x \cos x - 3 \cos x) \, dx$ 。 1M

$$\int \frac{(2 \sin x - 3) \cos x}{\sin^2 x - 3 \sin x + 2} \, dx = \int \frac{2 \sin x \cos x - 3 \cos x}{\sin^2 x - 3 \sin x + 2} \, dx$$

$$= \int \frac{du}{u}$$

1A

$$= \ln |\sin^2 x - 3 \sin x + 2| + \text{常數}$$

1A

12. (a) 設 $u = e^{3x} - 4$ 。則 $du = 3e^{3x} \, dx$ 。 1M

$$\int \frac{e^{3x}}{3^{3x} - 4} \, dx = \frac{1}{3} \int \frac{du}{u}$$

1A

$$= \frac{1}{3} \ln |e^{3x} - 4| + \text{常數}$$

1A

- (b) 設 $u = e^{2x} - 1$ 。則 $du = 2e^{2x} \, dx$ 。 1M

$$\int \frac{e^x}{e^x - e^{-x}} \, dx = \frac{1}{2} \int \frac{2e^{2x}}{e^{2x} - 1} \, dx$$

$$= \frac{1}{2} \int \frac{du}{u}$$

1A

$$= \frac{1}{2} \ln |e^{2x} - 1| + \text{常數}$$

1A

13. (a) $(1 + ax)^7 = \sum_{r=0}^7 C_r^7 a^r x^r$ 1M

$$\frac{\beta_1}{\beta_3} = \frac{C_1^7 a}{C_3^7 a^3} = \frac{1}{45}$$

1M

$$\frac{7}{35a^2} = \frac{1}{45}$$

$$a^2 = 9$$

$$a = 3 \quad \text{or} \quad -3 \quad (\text{捨去})$$

1A

- (b) $\int \sum_{r=0}^7 \beta_r x^r \, dx = \int (1 + 3x)^7 \, dx$ 1M

$$= \frac{(1 + 3x)^8}{24} + \text{常數}$$

1A

$$14. \quad (a) \quad \int \cos 4x \cos x \, dx = \frac{1}{2} \int (\cos 5x + \cos 3x) \, dx \quad 1M$$

$$= \frac{\sin 5x}{10} + \frac{\sin 3x}{6} + \text{常數} \quad 1A$$

$$(b) \quad \frac{\sin 6x - \sin 2x}{\sin x} = \frac{2 \cos 4x \sin 2x}{\sin x} \quad 1M$$

$$= \frac{2 \cos 4x (2 \sin x \cos x)}{\sin x} \quad 1M$$

$$= 4 \cos 4x \cos x \quad 1$$

$$(c) \quad \int \frac{\sin 6x}{\sin x} \, dx = \int \left(\frac{\sin 6x - \sin 2x}{\sin x} + \frac{\sin 2x}{\sin x} \right) \, dx \quad 1M$$

$$= 4 \int \cos 4x \cos x \, dx + \int \frac{\sin 2x}{\sin x} \, dx$$

$$= 4 \int \cos 4x \cos x \, dx + 2 \int \cos x \, dx \quad 1M$$

$$= 4 \left(\frac{\sin 5x}{10} + \frac{\sin 3x}{6} \right) + 2 \sin x + \text{常數} \quad 1M$$

$$= 2 \sin x + \frac{2 \sin 5x}{5} + \frac{2 \sin 3x}{3} + \text{常數} \quad 1A$$

$$15. \quad (a) \quad \int \cos^6 \frac{x}{2} \sin 2x \, dx = \int \left(\frac{1 + \cos x}{2} \right)^3 (2 \sin x \cos x) \, dx \quad 1M$$

$$\text{設 } u = \cos x \circ \text{ 則 } du = -\sin x \, dx \circ \quad 1M$$

$$\int \cos^6 \frac{x}{2} \sin 2x \, dx = -\frac{1}{4} \int (1 + u)^3 u \, du$$

$$= -\frac{1}{4} (u + 3u^2 + 3u^3 + u^4) \, du \quad 1A$$

$$= -\frac{1}{4} \left(\frac{\cos^2 x}{2} + \cos^3 x + \frac{3 \cos^4 x}{4} + \frac{\cos^5 x}{5} \right) + \text{常數}$$

$$= -\frac{\cos^2 x}{8} - \frac{\cos^3 x}{4} - \frac{3 \cos^4 x}{16} - \frac{\cos^5 x}{20} + \text{常數} \quad 1A$$

$$(b) \quad \text{設 } u = 2^x \circ \text{ 則 } du = 2^x \ln 2 \, dx \circ \quad 1M$$

$$\int 2^x \cos^6 2^{x-1} \sin 2^{x+1} \, dx = \frac{1}{\ln 2} \int \cos^6 \frac{u}{2} \sin 2u \, du \quad 1A$$

$$= \frac{1}{\ln 2} \left(\frac{\cos^2 u}{8} - \frac{\cos^3 u}{4} - \frac{3 \cos^4 u}{16} - \frac{\cos^5 u}{20} \right) + \text{常數} \quad 1A$$

$$= -\frac{\cos^2 2^x}{8 \ln 2} - \frac{\cos^3 2^x}{4 \ln 2} - \frac{3 \cos^4 2^x}{16 \ln 2} - \frac{\cos^5 2^x}{20 \ln 2} + \text{常數} \quad 1A$$

16. (a) $Ax + B + \frac{C}{x-4} + \frac{D}{x+2}$

$$= \frac{(Ax+B)(x-4)(x+2) + C(x+2) + D(x-4)}{(x-4)(x+2)}$$

故此， $2x^3 - 4x^2 - 13x \equiv (Ax+B)(x-4)(x+2) + C(x+2) + D(x-4)$ 。

$$\begin{cases} 2(4)^3 - 4(4)^2 - 13(4) = 6C & 1M \\ 2(-2)^3 - 4(-2)^2 - 13(-2) = -6D & \\ 0 = B(-4)(2) + 2C - 4D & \\ 2 - 4 - 13 = (A+B)(1-4)(1+2) + 3C - 3D & \end{cases}$$

求解後，可得 $A = 2$ 、 $B = 0$ 、 $C = 2$ 及 $D = 1$ 。

1A+1A

(b) $\int \frac{2x^3 - 4x^2 - 13x}{x^2 - 2x - 8} dx = \int \left(2x + \frac{2}{x-4} + \frac{1}{x+2} \right) dx$ 1M

$$= x^2 + 2 \ln|x-4| + \ln|x+2| + \text{常數}$$
 1A

(c) 設 $u = e^x$ 。則 $du = e^x dx$ 。 1M

$$\begin{aligned} \int \frac{2e^{2x} - 4e^x - 13}{8e^{-2x} + 2e^{-x} - 1} dx &= \int \frac{(2e^{3x} - 4e^{2x} - 13e^x)e^x}{-x^2 - 2e^x + 8} dx \\ &= - \int \frac{2u^3 - 4u^2 - 13u}{u^2 - 2u + 8} du & 1A \end{aligned}$$

$$= -u^2 - 2 \ln|u-4| - \ln|u+2| + \text{常數}$$

$$= -e^{2x} - 2 \ln|e^x - 4| - \ln|e^x + 2| + \text{常數}$$
 1A

17. (a) 設 $u = 6 \sin x - 5 \cos x$ 。則 $du = (6 \cos x + 5 \sin x) dx$ 。 1M

$$\begin{aligned} \int \frac{5 \sin x + 6 \cos x}{6 \sin x - 5 \cos x} dx &= \int \frac{du}{u} \\ &= \ln|6 \sin x - 5 \cos x| + \text{常數} & 1A \end{aligned}$$

(b) (i) $af(x) + bg(x) = (5a + 6b) \sin x + (6a - 5b)g(x)$

可得 $5a + 6b = 1$ 及 $6a - 5b = -1$ 。 1M

求解後，可得 $a = -\frac{1}{61}$ 及 $b = \frac{11}{61}$ 。 1A

因此， $\sin x - \cos x = -\frac{1}{61}f(x) + \frac{11}{61}g(x)$ 。

(ii) $\int \frac{\sin x - \cos x}{6 \sin x - 5 \cos x} dx = -\frac{1}{61} \int \frac{5 \sin x + 6 \cos x}{6 \sin x - 5 \cos x} dx + \frac{11}{61} \int \frac{6 \sin x - 5 \cos x}{6 \sin x - 5 \cos x} dx$ 1M

$$= -\frac{1}{61} \int \frac{5 \sin x + 6 \cos x}{6 \sin x - 5 \cos x} dx + \frac{11}{61} \int dx$$
 1M

$$= -\frac{1}{61} \ln|6 \sin x - 5 \cos x| + \frac{11x}{61} + \text{常數}$$
 1A