Dexter Wong & His Mathematics Team

暑期課程　2022 – 2023

數學　延伸部分

S5 – S6 M2 微分　功課 $\boxed{\text{第 2 套}}$

名字：________________　　電話：________

學生編號：________________　　分校：________

上課日：日 / 一 / 二 / 三 / 四 / 五 / 六

考生須知

1. 本試卷**各題均須作答**，答案須寫在本試題答題簿中預留的空位內。
2. 除特別指明外，須詳細列出所有算式。
3. 除特別指明外，數值答案須用真確值表示。
4. 本試卷的附圖不一定依比例繪成。

建議題解

派發於暑期課程 S5 – S6 M2 微分 第 1 期 – 第 2 堂

1. (a) $y = \dfrac{(x+4)(x-3)^2}{(x+1)^3}$

$= \dfrac{(u+3)(u-4)^2}{u^3}$ 1A

(b) $y = \dfrac{u^3 - 5u^2 - 8u + 48}{u^3}$

$= 1 - 5u^{-1} - 8u^{-2} + 48u^{-3}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ 1M

$= \left[5u^{-2} + 16u^{-3} - 144u^{-4}\right](1)$ 1A

$= \dfrac{5}{u^2} + \dfrac{16}{u^3} - \dfrac{144}{u^4}$

$= \dfrac{5}{(x+1)^2} + \dfrac{16}{(x+1)^3} - \dfrac{144}{(x+1)^4}$ 1A

2. (a) $y = \dfrac{(3x-1)^2(4x+3)}{(2x+1)^4}$

$= \dfrac{\left[3\left(\frac{u-1}{2}\right) - 1\right]^2 \left[4\left(\frac{u-1}{2}\right) + 3\right]}{u^4}$ 1A

$= \dfrac{(3u-5)^2(2u+1)}{4u^2}$ 1A

(b) $y = \dfrac{18u^3 - 51u^2 + 20u + 25}{4u^4}$

$= \dfrac{9}{2}u^{-1} - \dfrac{51}{4}u^{-2} + 5u^{-3} + \dfrac{25}{4}u^{-4}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ 1M

$= \left(-\dfrac{9}{2u^2} + \dfrac{51}{2u^3} - \dfrac{15}{u^4} - \dfrac{25}{u^5}\right)(2)$ 1A

$= -\dfrac{9}{(2x+1)^2} + \dfrac{51}{(2x+1)^3} - \dfrac{30}{(2x+1)^4} - \dfrac{50}{(2x+1)^5}$ 1

3. (a) $y = \dfrac{(2x-3)^3}{(x+5)^4}$

$= \dfrac{[2(u-5)-3]^3}{u^4}$

$= \dfrac{(2u-13)^3}{u^4}$ 1A

(b) $y = \dfrac{(2u-13)^3}{u^4}$

$= \dfrac{8u^3 - 156u^2 + 1014u - 2197}{u^4}$

$= 8u^{-1} - 156u^{-2} + 1014u^{-3} - 2197u^{-4}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ 1M

$= \left(-8u^{-2} + 312u^{-3} - 3042u^{-4} + 8788u^{-5}\right)(1)$ 1A

$= -8u^{-2} + 312u^{-3} - 3042u^{-4} + 8788u^{-5}$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{du}\left(-8u^{-2} + 312u^{-3} - 3042u^{-4} + 8788u^{-5}\right) \cdot \dfrac{du}{dx}$ 1M

$= 16u^{-3} - 936u^{-4} + 12\,168u^{-5} - 43\,940u^{-6}$

$= \dfrac{16}{(x+5)^3} - \dfrac{936}{(x+5)^4} + \dfrac{12\,168}{(x+5)^5} - \dfrac{43\,940}{(x+5)^6}$ 1

4. (a) $y = \dfrac{(6x+5)^3}{(3x+1)^2}$

$= \dfrac{[2(u-1)+5]^3}{u^2}$

$= \dfrac{(2u+3)^3}{u^2}$ 1A

(b) $y = \dfrac{8u^3 + 36u^2 + 54u + 27}{u^2}$

$= 8u + 36 + 54u^{-1} + 27u^{-2}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ 1M

$= \left(8 - 54u^{-2} - 54u^{-3}\right)(3)$ 1A

$= 24 - 162u^{-2} - 162u^{-3}$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(24 - 162u^{-2} - 162u^{-3}\right)\dfrac{du}{dx}$

$= \left(324u^{-3} + 486u^{-4}\right)(3)$ 1A

$= 972u^{-3} + 1458u^{-4}$

$= \dfrac{972u + 1458}{u^4}$

$= \dfrac{972(3x+1) + 1458}{(3x+1)^4}$

$= \dfrac{486(6x+5)}{(3x+1)^4}$ 1A

5. (a) $x^2 + y^2 = 30$

$$2x + 2y\frac{dy}{dx} = 0 \qquad \text{1M}$$

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \text{1A}$$

$$\left.\frac{dy}{dx}\right|_{(p,\ q)} = \frac{-p}{q} = -\frac{1}{2}$$

$$q = 2p \qquad 1$$

(b) $p^2 + (2p)^2 = 30$ 1M

$$p^2 = 6$$

$$p = \pm\sqrt{6}$$

因此，$(p,\ q) = \left(\sqrt{6},\ 2\sqrt{6}\right)$ 或 $\left(-\sqrt{6},\ -2\sqrt{6}\right)$。 1A+1A

6. $$\frac{d}{dx}[\sin x + \sin 3x + \ldots + \sin(2n-1)x] = \frac{d}{dx}\left[\frac{1-\cos 2nx}{2\sin x}\right] \qquad \text{1M}$$

$$\cos x + 3\cos 3x + \ldots + (2n-1)\cos(2n-1)x = \frac{(2n\sin 2nx)(\sin x) - (1-\cos 2nx)(\cos x)}{2\sin^2 x} \qquad \text{1M}$$

代 $n = 18$ 及 $x = \frac{\pi}{6}$，

$$\cos\frac{\pi}{6} + 3\cos\frac{3\pi}{6} + \ldots + 35\cos\frac{35\pi}{6}$$

$$= \frac{36\sin(6\pi)\left(\sin\frac{\pi}{6}\right) - (1-\cos 6\pi)\left(\cos\frac{\pi}{6}\right)}{2\sin^2\frac{\pi}{6}} \qquad \text{1M}$$

$$= \frac{0-0}{2\left(\frac{1}{2}\right)^2}$$

$$= 0 \qquad \text{1A}$$

7. (a)

$$\begin{aligned}
y &= \frac{\sin^2 x + \csc^2 x}{\sin^2 x - \csc^2 x} \\
&= \frac{u + \frac{1}{u}}{u - \frac{1}{u}} \times \frac{u}{u} && \text{1M} \\
&= \frac{u^2 + 1}{u^2 - 1} \\
&= 1 + \frac{2}{u^2 - 1} \\
\frac{\mathrm{d}y}{\mathrm{d}u} &= 2(u^2 - 1)^{-2}(2u) && \text{1M} \\
&= \frac{4u}{(u^2 - 1)^2} && \text{1A}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} && \text{1M} \\
&= -\frac{4u}{(u^2 - 1)^2} \cdot (2\sin x \cos x) \\
&= -\frac{4\sin^2 x(2\sin x \cos x)}{(\sin^2 x + 1)^2(\sin^2 x - 1)^2} \\
&= -\frac{8\sin^3 x \cos x}{(\sin^2 x + 1)^2(-\cos^2 x)^2} \\
&= -\frac{8\tan^3 x}{(\sin^2 x + 1)^2} && \text{1}
\end{aligned}$$

8.

$$\begin{aligned}
\sin xy &= \sec(x + y)^2 \\
\sin xy &= [\cos(x + y)^2]^{-1} && \text{1M} \\
\cos xy \left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= -[\cos(x + y)^2]^{-2}[-\sin(x + y)^2](2)(x + y)\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) && \text{1M} \\
\left[x\cos xy - 2(x + y)\sin(x + y)^2 \sec^2(x + y)^2\right]\frac{\mathrm{d}y}{\mathrm{d}x} &= 2(x + y)\sin(x + y)^2 \sec^2(x + y)^2 - y\cos xy \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(x + y)\sin(x + y)^2 \sec^2(x + y)^2 - y\cos xy}{x\cos xy - 2(x + y)\sin(x + y)^2 \sec^2(x + y)^2} && \text{1A}
\end{aligned}$$

9.

$$\begin{aligned}
\sin(x^2 + y) &= \tan(x + 1) - 1 \\
\cos(x^2 + y)\left(2x + \frac{\mathrm{d}y}{\mathrm{d}x}\right) &= \sec^2(x + 1) && \text{1M} \\
\cos(x^2 + y)\frac{\mathrm{d}y}{\mathrm{d}x} &= \sec^2(x + 1) - 2x\cos(x^2 + y) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\sec^2(x + 1) - 2x\cos(x^2 + y)}{\cos(x^2 + y)} && \text{1A}
\end{aligned}$$

10. 留意在該函數的定義域中，可得 $\cot x^2 > 0$，即 $\tan x^2 > 0$。

$$y = \left[\ln\left(\tan^2 x^2\right)\right]\left(\ln\sqrt{\cot x^2}\right)$$
$$= 2\ln\left(\tan x^2\right)\cdot\left(-\frac{1}{2}\right)\ln\left(\tan x^2\right)$$
$$= -\left[\ln\left(\tan x^2\right)\right]^2 \qquad \text{1A}$$
$$\frac{dy}{dx} = -2\left[\ln\left(\tan x^2\right)\right]\left(\frac{1}{\tan x^2}\right)(\sec^2 x^2)(2x) \qquad \text{1M}$$
$$= -4x\csc x^2 \sec x^2 \ln\left(\tan x^2\right) \qquad \text{1A}$$

11.
$$\ln\left(x^2 y\right) = 4x^3 - 10y$$
$$\frac{1}{x^2 y}\left(2xy + x^2\frac{dy}{dx}\right) = 12x^2 - 10\frac{dy}{dx} \qquad \text{1M}$$
$$\left(\frac{1}{y} + 10\right)\frac{dy}{dx} = 12x^2 - \frac{2}{x}$$
$$\left(\frac{1+10y}{y}\right)\frac{dy}{dx} = \frac{12x^3 - 2}{x}$$
$$\frac{dy}{dx} = \frac{2y(6x^3 - 1)}{x(1+10y)} \qquad \text{1A}$$

12.
$$e^{3x-2y} + x = y - \cos y$$
$$e^{3x-2y}\left(3 - 2\frac{dy}{dx}\right) + 1 = \frac{dy}{dx} + \sin y\frac{dy}{dx} \qquad \text{1M}$$
$$\left(-2e^{3x-2y} - 1 - \sin y\right)\frac{dy}{dx} = -3e^{3x-2y} - 1$$
$$\frac{dy}{dx} = \frac{3e^{3x-2y} + 1}{2e^{3x-2y} + 1 + \sin y} \qquad \text{1A}$$

13.
$$y' = x^k\left(\frac{1}{x}\right) + kx^{k-1}\ln x$$
$$= x^{k-1}(1 + k\ln x) \qquad \text{1A}$$
$$y'' = (k-1)x^{k-2}(1 + k\ln x) + x^{k-1}\left(\frac{k}{x}\right)$$
$$= x^{k-2}[2k - 1 + k(k-1)\ln x] \qquad \text{1A}$$
$$4x^2y'' - 8xy' + 9y = 0$$
$$4x^k[2k - 1 + k(k-1)\ln x] - 8x^k(1 + k\ln x) + 9x^k\ln x = 0$$
$$x^k\left[(4k^2 - 12k + 9)\ln x + 8k - 12\right] = 0$$
$$x^k\left[(2k-3)^2\ln x + 4(2k-3)\right] = 0$$
$$x^k(2k-3)\left[(2k-3)\ln x + 4\right] = 0 \qquad \text{1M}$$
$$2k - 3 = 0$$
$$k = \frac{3}{2} \qquad \text{1A}$$

14. (a) $y^2 = x + x^{\frac{1}{2}}$

$2yy' = 1 + \frac{1}{2}x^{-\frac{1}{2}}$ 1M

$4yy' - x^{-\frac{1}{2}} = 2$ 1

(b) $4yy' - x^{-\frac{1}{2}} = 2$

$4(y')^2 + 4yy'' + \frac{1}{2}x^{-\frac{3}{2}} = 0$ 1M

$yy'' + (y')^2 + \frac{1}{8}x^{-\frac{3}{2}} = 0$ 1

15. (a) $y = x^2e^x$

$y' = 2xe^x + x^2e^x$ 1A

$y'' = (2e^x + 2xe^x) + (2xe^x + x^2e^x)$

$= 2e^x + 4xe^x + x^2e^x$ 1A

(b) $y'' + y' - 2y = (2e^x + 4xe^x + x^2e^x) + (2xe^x + x^2e^x) - 2x^2e^x$ 1M

$= e^x[2 + (4x + 2x) + (x^2 + x^2 - 2x^2)]$

$= e^x(6x + 2)$

因此，$a = 6$ 及 $b = 2$。 1A+1A

16. $\frac{dy}{dx} = 2n(2x + 3)^{n-1}$ 1A

$\frac{d^2y}{dx^2} = 4n(n - 1)(2x + 3)^{n-2}$ 1A

由於對所有實數 x，$(2x + 3)^2\frac{d^2y}{dx^2} - 9(2x + 3)\frac{dy}{dx} + 10y = 0$，

$4n(n - 1)(2x + 3)^n - 18n(2x + 3)^n + 10(2x + 3)^n = 0$ 1M

$(2x + 3)^n(4n^2 - 22n + 10) = 0$

$n = 5$ 或 $\frac{1}{2}$（捨去） 1A